

ON THE ELLIPTIC \mathfrak{gl}_2 SOLID-ON-SOLID MODEL: FUNCTIONAL RELATIONS AND DETERMINANTS

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ABSTRACT. In this work we study an elliptic solid-on-solid model with domain-wall boundaries having the elliptic quantum group $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$ as its underlying symmetry algebra. We elaborate on results previously presented in [Gal16a] and extend our analysis to include continuous families of single determinantal representations for the model's partition function. Interestingly, our families of representations are parameterized by two continuous complex variables which can be arbitrarily chosen without affecting the partition function.

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1. INTRODUCTION

Integrable systems have a long history of introducing new concepts and several developments in physics and mathematics can be credited to their study. Among the list of developments triggered by the study of integrable systems *Quantum Groups* have a special place. The formulation of Quantum Groups is one of the main achievements of the *Quantum Inverse Scattering Method* [STF79, TF79] and its origin is intimately related to the problem of finding solutions of the Yang-Baxter equation. Such solutions are usually referred to as \mathcal{R} -matrices and they are central objects within the theory of *quantum integrable systems*. In addition to that, exactly solvable vertex models of Statistical Mechanics also allow for a formulation having their statistical weights encoded in a \mathcal{R} -matrix.

The six-vertex model has played an important role in this development and the constructions nowadays known as *Yangian* [Dri85] and *q-deformation* [KR83] firstly appeared associated to the six-vertex model \mathcal{R} -matrix [Lie67, Sut67]. In a more general setting, for each simple finite-dimensional Lie algebra \mathfrak{g} , the Hopf algebra Yangian $\mathcal{Y}[\mathfrak{g}]$ produces a rational solution of the Yang-Baxter equation while the *q*-deformation $\mathcal{U}_q[\mathfrak{g}]$ yields a trigonometric one. However, \mathcal{R} -matrices with rational and/or trigonometric entries do not exhaust all possible solutions of the Yang-Baxter equation; and a distinguished role is played by elliptic solutions. For instance, the eight-vertex model is a corner stone of the theory of exactly solvable models of Statistical Mechanics and its statistical weights are parameterized by elliptic functions [Bax71].

On the other hand, among the list of solvable two-dimensional lattice models we also have the so called solid-on-solid models. They are closely related to the eight-vertex model and also play a prominent role in Statistical Mechanics. The statistical weights of solid-on-solid models are also parameterized by elliptic functions and the algebraic structure ensuring integrability of such models are nowadays known as *Elliptic Quantum Groups*.

1.1. Solid-on-solid models. The introduction of solid-on-solid models, or SOS models for short, is intimately related to the study of Baxter's eight-vertex model [Bax73]. The

former was originally put forward as a tool for describing eigenvectors of the symmetric eight-vertex model but it has gained life on its own. In the literature they are also referred to as *interaction-round-a-face* models and their collective interactions are characterized by variables assigned to the lattice sites instead of edges; the latter being the usual description employed in vertex models.

The dimensionality of a lattice model also plays a fundamental role for establishing its integrability in the sense of Baxter [Bax07]; and here we restrict our discussion to models defined on a two-dimensional lattice. In addition to that, several equivalences between two-dimensional lattice systems of Statistical Mechanics have been established over the years and it is worth remarking that the relation between SOS and vertex models configurations was firstly noticed by Lenard [Lie67, Note added in proof]. Further relations have been unveiled over the years and, for instance, one also finds that the SOS model associated to Baxter's eight-vertex model consists of an Ising-type model with four-spin interactions [Bax07]. For a detailed discussion on the formulation of SOS models we refer the reader to [Bax07, DMR85] and references therein.

As far as the eight-vertex model is concerned, the algebraic structure underlying Baxter's elliptic uniformization consists of Sklyanin algebra [Sk182, Sk183]. On the other hand, despite the close connection between the elliptic SOS model considered in the present paper and Baxter's eight-vertex model, the algebraic structure underlying SOS models are not captured by Sklyanin algebras. For SOS models one needs to invoke the concept of Elliptic Quantum Groups which was put forward by Felder in [Fel94, Fel95].

1.2. Elliptic quantum groups. The elliptic nature of Baxter's eight-vertex model arises from the requirement that the model's statistical weights satisfy the Yang-Baxter equation [Bax71, Bax72]. On the other hand, there exists a remarkable relation between the symmetric eight-vertex model and a particular SOS model [Bax73]. Baxter's *vertex-face transformation* precises this relation and the elliptic nature of the eight-vertex model is consequently transported to the dual SOS model. In fact, the vertex-face transformation of Baxter's eight-vertex model endows the resulting SOS model with an additional continuous parameter which will be later on referred to as *dynamical parameter*. As far as integrability in the sense of Baxter is concerned, the Yang-Baxter equation for SOS models takes the form of the so called *hexagon identity* [DMR85]. The role played by the hexagon identity for SOS models is exactly the same as the one played by the standard Yang-Baxter equation for vertex models: it ensures the model's transfer matrix forms a commuting family.

The algebraic structures underlying integrable vertex models mostly consist of Drinfel'd-Jimbo quantum enveloping algebras [Dri85, Dri87, Jim85, Jim86], Yangians [Dri85, Dri87] and Sklyanin algebras [Sk182, Sk183]. As for SOS models an analogous structure was only unveiled in [Fel94, Fel95] by Felder and it received the name Elliptic Quantum Groups. In [Fel94, Fel95] Felder also showed that statistical weights satisfying the hexagon identity are encoded in solutions of a *dynamical* version of the Yang-Baxter equation. This dynamical equation was proposed as a quantization of a modified classical Yang-Baxter equation arising as the compatibility condition for the Knizhnik-Zamolodchikov-Bernard equation [KZ84, Ber88b, Ber88a]. It is worth remarking here that the dynamical Yang-Baxter

equation; however, without spectral parameter, had previously appeared in the work of Gervais and Neveu on the Liouville string field theory [GN84]. Elliptic Quantum Groups can be defined for any simple Lie algebra \mathfrak{g} and they are usually denoted by $\mathcal{E}_{p,\gamma}[\mathfrak{g}]$. They provide algebraic foundations for solutions of the dynamical Yang-Baxter equation and in the present paper we shall restrict our discussion to the case $\mathfrak{g} = \mathfrak{gl}_2$.

1.3. Domain-wall boundaries. Three main ingredients are required in order to define a vertex or SOS model: graphs on a lattice, statistical weights for graphs configurations and boundary conditions. Integrability in the sense of Baxter imposes restriction on all of them and, in particular, inappropriate choices of boundary conditions can render the model trivial or break its bulk integrability. The Yang-Baxter equation and its dynamical version govern the model's bulk integrability, while integrability preserving boundary conditions are usually singled out from additional set of constraints. See for instance [dV84] and [Sk188]. However, there still exist certain types of boundary conditions rendering the model exactly solvable which are not characterized by extra algebraic equations. This is precisely the case of domain-wall boundary conditions introduced by Korepin in order to study scalar products of Bethe vectors for the six-vertex model [Kor82]. Such boundary conditions render a well defined system of Statistical Mechanics and, interestingly, the partition function of the six-vertex model with domain-wall boundaries has been exactly computed in a closed form. It can be expressed as determinants [Ize87, CP08, Gal16b] and multiple contour integrals [dGGS11, Gal12, Gal13b]; in contrast to the case with toroidal boundary conditions whose partition function evaluation still depends on the resolution of Bethe ansatz equations [Lie67].

Boundary conditions of domain-wall type have also been considered for SOS models, see for instance [RS09a, RS09b, PRS08, Ros09, Gal12, Gal13b]. In particular, the model's partition function was shown in [RS09a] to satisfy a rather simple functional equation when the anisotropy parameter satisfies a root-of-unity condition. For generic values of the anisotropy, we have put forward different kinds of functional relations having their origins in the dynamical Yang-Baxter algebra [Gal12, Gal13b]. The resolution of those functional equations resulted in a multiple contour integral representation for the model's partition function for generic values of the anisotropy parameter. Furthermore, the authors of [PRS08] have shown that the partition function of the elliptic SOS model with domain-wall boundaries consists of an *universal elliptic weight function*. The latter relation has also led to a sum over permutations representation for the model's partition function. As far as determinantal representations are concerned, in the work [Ros09] Rosengren has presented a sum of Frobenius type determinants which seems to generalize Izergin's representation for the six-vertex model [Ize87]. More recently, Rosengren has also shown that the aforementioned partition function can be written as the sum of two *pfaffians* when the model's anisotropy parameter is fixed to a particular root-of-unity value [Ros16].

It is worth remarking that SOS models with domain-walls and one reflecting end have also been studied in the literature. Interestingly, in that case the model's dynamical structure does not offer obstacles for writing down the model's partition function as a single determinant [FK10, Fil11]. Functional equations describing the latter partition function are

also available and they have been obtained in [Lam15] generalizing the results of [GL14] for the trigonometric (non-dynamical) model. The resolution of the aforementioned functional equations has produced multiple contour integral representations for the corresponding partition functions.

1.4. Applications. Vertex models were proposed by Pauling in 1935 in order to explain the ice residual entropy [Pau35] and several other applications have emerged over the years. In their turn, solid-on-solid models enjoy a similar status and some applications involve the description of physical systems whilst others have a purely mathematical scope. As far as physical applications are concerned, the case with periodic boundary conditions fills most of the literature up to the present days. For instance, the case known as *restricted* SOS model [ABF84, FB85] was shown to realize certain unitary minimal models in the continuum scaling limit [Hus84]. Also, the restricted SOS model is known to describe non-local statistics of height clusters and percolation hull exponents [SD87].

Physical applications of the elliptic SOS model with domain-wall boundary conditions are still rather limited and this is possibly due to the lack of appropriate representations for the model's partition function. However, this model exhibits a very rich structure and interesting mathematical applications have been found over the past years. The partition function of the elliptic SOS model with domain-wall boundaries has been studied through several approaches, see for instance [PRS08, Ros09, Gal13b], and remarkable properties have been reported when the model degenerates into the so-called three-color model. The latter is obtained by fixing the model's anisotropy parameter as a particular root-of-unity value coinciding with Kuperberg's specialization used for the six-vertex model [Kup96]. The trigonometric six-vertex model under Kuperberg's specialization is known to enumerate *Alternating Sign Matrices* (ASM) and a similar analysis for the three-color model has been performed in [Ros09]. In that case Rosengren has found that, besides the enumeration of ASM, the three-color model also counts the number of *Cyclically Symmetric Plane Partitions* (CSPP).

In addition to applications in enumerative combinatorics the three-color model with domain-wall boundaries has also found applications in the theory of *special functions*. For instance, the work [Ros11] demonstrates that the aforementioned model gives rise to a family of two-variables orthogonal polynomials satisfying special recursion relations. Using these results Rosengren has conjectured an explicit formula for the model's free-energy in the thermodynamical limit [Ros11]. Other degeneration of the elliptic SOS model also exhibit special properties. The so-called *supersymmetric point* is one of them and, at this point, the model's partition function gives rise to certain symmetric polynomials whose properties have been studied in the series of works [Ros13a, Ros13b, Ros14, Ros15].

1.5. Algebraic-functional approach. Functional methods are at the core of the modern theory of integrable systems and the functional equation derived in [Gal13b] characterizing the partition function of the elliptic SOS model with domain-wall boundaries has a singular origin: the dynamical Yang-Baxter algebra. In particular, the equation of [Gal13b] exhibits similarities with Knizhnik-Zamolodchikov equations and this feature has been already discussed in [Gal13a]. Multiple contour integrals are known to accommodate solutions of

Knizhnik-Zamolodchikov equations [TV97, Var, EFK98] and this feature has motivated the search for similar representations for the solution of our functional equation.

As far as the origin of our equation is concerned, the possibility of exploiting the Yang-Baxter algebra and its dynamical version in such a manner did not appear for the first time in [Gal13b]. It was firstly put forward in [Gal08] for spectral problems and in [Gal10] for partition functions with special boundary conditions. Scalar products of Bethe vectors have been tackled through this method as well, see [Gal14, Gal15a], resulting in compact representations for *off-shell* scalar products. Furthermore, in [Gal11, Gal13a] we have also pointed out the existence of families of *Partial Differential Equations* underlying such functional relations. The mechanism leading to such differential equations have been further developed in the works [Gal15b, GL14, GL15] which includes the analysis of the toroidal six-vertex model spectrum and *on-shell* scalar products of Bethe vectors. On the other hand, an alternative mechanism leading to a system of first-order partial differential equations has been recently described in [Gal16b]. Although the results of [Gal16b] are restricted to the rational six-vertex model with domain-wall boundaries, whose functional equation follows from a particular limit of the one presented in [Gal13b], the generalization of [Gal16b] to the trigonometric case is straightforward.

The system of partial differential equations found in [Gal16b] exhibits uncanny similarities with classical Knizhnik-Zamolodchikov equations and its resolution has produced a discrete family of single determinant representations for the six-vertex model's partition function. The representations found in [Gal16b] do not seem to be reminiscent from Izergin's determinant [Ize87] and, by way of contrast, they offer trivial access to the model's partial homogeneous limit.

The algebraic structures underlying quantum integrable systems are not limited to the Yang-Baxter algebra and its dynamical version. For instance, integrable systems with open boundary conditions also obey the so called *reflection algebra*. The latter algebraic structure can also be exploited along the lines of the algebraic-functional framework and this possibility was demonstrated in [GL14]. Moreover, this approach is also feasible using a dynamical version of the reflection algebra as shown by Lamers in [Lam15].

1.6. The goals of this paper. Integrable systems usually share common structures and one relevant question which has been debated over the past years is if the partition function of the elliptic SOS model with domain-wall boundaries can be accommodated by a single determinant formula. As a matter of fact, the results presented in the literature so far suggested that the answer for this question was negative. For instance, the representation found by Rosengren in [Ros09] is given in terms of a sum of determinants which reduces to Izergin's single determinant formula in the six-vertex model limit. This was actually a good indication that Rosengren's formula is indeed the natural extension of [Ize87] and, therefore, single determinant representations should not exist.

However, the family of representations recently found in [Gal16b] do not share any similarity with Izergin's formula, although they are also expressed as a single determinant. In addition to that, the method employed in [Gal16b] does not exhibit any overlap with the recursive approach of Korepin [Kor82] which ultimately leads to the results of [Ize87].

Given the scenario above described it is natural to ask if the approach proposed in [Gal16b] can be extended to the elliptic SOS model. Moreover, if that is indeed possible, will the solution be given by a single determinant? The answer for this question is positive and the derivation of such representations along the lines of the Algebraic-Functional framework is what we intend to discuss in the present work.

Outline. We have organized this paper placing the spotlight on the approach leading to our main result. The formulation of the model under consideration has been already extensively discussed in the literature and we shall not go into those details. The mathematical definitions and conventions employed throughout this work can be found in Section 2. The main ideas of the Algebraic-Functional (AF) method are presented in Section 3, although some technical aspects of the AF method are omitted since they have been discussed in previous works. On the other hand, generalizations required to obtain the announced determinantal representations will be discussed in some detail. The functional relation used to obtain our determinantal representations is derived in Section 4 and its resolution is discussed in Section 5. We leave Section 6 for the analysis of the six-vertex model limit and the proofs of our main theorems are given in Appendix A and Appendix B. For the sake of clarity, we gather explicit formulae required for building our determinantal representations in Appendix C.

2. DEFINITIONS AND CONVENTIONS

This work is concerned with a particular two-dimensional lattice model of Statistical Mechanics, alias elliptic SOS model, whose statistical weights are intimately related to the elliptic quantum group $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$. This model has been extensively discussed in the literature and, in this way, we shall restrict ourselves to presenting only the mathematical definitions required throughout this work. The model definition includes statistical weights for plaquette's configurations and boundary conditions characterizing the relevant partition function. As far as those ingredients are concerned, we shall mostly use the conventions of [Gal12, Gal13b]. Statistical weights of integrable SOS models can be encoded in solutions of the dynamical Yang-Baxter equation [Fel94, Fel95] and this will be the starting point of our discussion.

2.1. Dynamical Yang-Baxter equation. Let \mathcal{V} be a complex vector space and $\mathfrak{gl}(\mathcal{V})$ the associated Lie algebra. Here we shall use \mathfrak{h} to denote the Cartan subalgebra of $\mathfrak{gl}(\mathcal{V})$ which is also endowed with a symmetric bilinear form $(\cdot, \cdot): \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C}$. Moreover, let \mathcal{W} be a finite dimensional representation of \mathcal{V} and $\pi: \mathfrak{gl}(\mathcal{V}) \rightarrow \mathfrak{gl}(\mathcal{W})$ be the representation map. In addition to that we also define the weight module $\mathcal{W}[\lambda]$ as

$$(2.1) \quad \mathcal{W}[\lambda] := \{w \in \mathcal{W} \mid \pi(h)w = (h, \lambda)w \quad \forall h \in \mathfrak{h}\}.$$

In (2.1) $\lambda \in \mathfrak{h}$ is referred to as weight of \mathcal{W} if $\mathcal{W}[\lambda]$ is non-empty; and we write $\Lambda(\mathcal{W}) \subset \mathfrak{h}$ for the set of weights of \mathcal{W} . Then, as representation of $\mathfrak{gl}(\mathcal{V})$, \mathcal{W} admits the weight

decomposition

$$(2.2) \quad \mathcal{W} = \bigoplus_{\lambda \in \Lambda(\mathcal{W})} \mathcal{W}[\lambda] .$$

Next we suppose $\mathcal{Q}(\zeta): \mathcal{W}^{\otimes n} \rightarrow \mathcal{W}^{\otimes n}$ is a linear operator depending meromorphically on $\zeta \in \mathfrak{h}$. We refer to $\mathcal{Q}(\zeta)$ as dynamical operator when it acts on $\mathcal{W}^{\otimes n}$ according to the direct sum decomposition

$$(2.3) \quad \mathcal{W}^{\otimes n} = \bigoplus_{\lambda \in \Lambda(\mathcal{W})} \mathcal{W}^{\otimes(i-1)} \otimes \mathcal{W}[\lambda] \otimes \mathcal{W}^{\otimes(n-i)} .$$

More precisely, for $\gamma \in \mathbb{C}$ and $1 \leq i \leq n$, we write $\mathcal{Q}(\zeta + \gamma h_i)$ for the operator acting as $\mathcal{Q}(\zeta + \gamma \lambda)$ on the projected subspace $\mathcal{W}^{\otimes(i-1)} \otimes \mathcal{W}[\lambda] \otimes \mathcal{W}^{\otimes(n-i)}$ of $\mathcal{W}^{\otimes n}$.

Now let $\mathcal{R}(x; \zeta) \in \text{End}(\mathcal{W}^{\otimes 2})$ be a dynamical linear operator whose dependence on $x \in \mathbb{C}$ and $\zeta \in \mathfrak{h}$ is meromorphic. Then, using the standard tensor leg notation, the dynamical Yang-Baxter equation is a relation in $\text{End}(\mathcal{W}^{\otimes 3})$ reading

$$(2.4) \quad \begin{aligned} &\mathcal{R}_{12}(x_1 - x_2; \tau - \gamma h_3) \mathcal{R}_{13}(x_1 - x_3; \tau) \mathcal{R}_{23}(x_2 - x_3; \tau - \gamma h_1) = \\ &\mathcal{R}_{23}(x_2 - x_3; \tau) \mathcal{R}_{13}(x_1 - x_3; \tau - \gamma h_2) \mathcal{R}_{12}(x_1 - x_2; \tau) , \end{aligned}$$

for $x_i, \tau, \gamma \in \mathbb{C}$.

2.2. The $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$ dynamical \mathcal{R} -matrix. The specialization of (2.4) to the \mathfrak{gl}_2 case is obtained by setting $\mathcal{V} = \mathbb{C}v_1 \oplus \mathbb{C}v_2$ with basis vectors $v_i \in \mathbb{C}^2$. Next we identify $\mathfrak{gl}(\mathcal{V})$ with a matrix algebra via the ordered basis $\{v_1, v_2\} \in \mathcal{V}$ in such a way that $\mathfrak{gl}(\mathcal{V}) \simeq \mathfrak{gl}_2$. The Cartan subalgebra is then $\mathfrak{h} := \mathbb{C}E_{11} \oplus \mathbb{C}E_{22}$ with matrix units $E_{ij} \in \mathfrak{gl}(\mathcal{V})$ defined through the action $E_{i,j}(v_k) := \delta_{j,k}v_i$ for $i, j, k \in \{1, 2\}$. Furthermore, the symmetric bilinear form $(\cdot, \cdot): \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C}$ is then given by

$$(2.5) \quad (E_{ii}, E_{jj}) = \begin{cases} 1 & \text{if } i = j = 1 \\ -1 & \text{if } i = j = 2 \\ 0 & \text{otherwise} \end{cases} .$$

For later convenience we also fix the elliptic nome $0 < p < 1$ and introduce the short hand notation

$$(2.6) \quad [x] := \frac{1}{2} \sum_{n=-\infty}^{+\infty} (-1)^{n-\frac{1}{2}} p^{(n+\frac{1}{2})^2} e^{-(2n+1)x}$$

for $x \in \mathbb{C}$.

Remark 2.1. We stress here that $[x]$ corresponds to the Jacobi theta-function $\Theta_1(ix, \nu)$ with $p = e^{i\pi\nu}$ according to the conventions of [WW27].

Given the above definitions we fix the parameter $\gamma \in \mathbb{C}$ in such a way that the solution of (2.4) associated to the elliptic quantum group $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$ explicitly reads

$$(2.7) \quad \mathcal{R}(x; \tau) = \begin{pmatrix} a_+(x, \tau) & 0 & 0 & 0 \\ 0 & b_+(x, \tau) & c_+(x, \tau) & 0 \\ 0 & c_-(x, \tau) & b_-(x, \tau) & 0 \\ 0 & 0 & 0 & a_-(x; \tau) \end{pmatrix}$$

with non-null entries defined by

$$(2.8) \quad \begin{aligned} a_{\pm}(x, \tau) &:= [x + \gamma] \\ b_{\pm}(x, \tau) &:= [\tau \mp \gamma][x][\tau]^{-1} \\ c_{\pm}(x, \tau) &:= [\tau \mp x][\gamma][\tau]^{-1}. \end{aligned}$$

The construction of (2.7) relies on the representation theory of the elliptic quantum group $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$. The latter has been described in [FV96].

2.3. Modules over $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$. Following [Fel94, Fel95] we refer to the algebra associated with the dynamical \mathcal{R} -matrix (2.7) as elliptic quantum group $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$. We shall also employ $\mathcal{A}(\mathcal{R})$ to denote this algebra and it is generated by meromorphic functions on \mathfrak{h} and non-commutative matrix elements of $\mathcal{L}(x, \tau) \in \text{End}(\mathcal{V})$ subjected to the following relation

$$(2.9) \quad \begin{aligned} \mathcal{R}_{12}(x_1 - x_2; \tau - \gamma h) \mathcal{L}_1(x_1, \tau) \mathcal{L}_2(x_2, \tau - \gamma h_1) = \\ \mathcal{L}_2(x_2, \tau) \mathcal{L}_1(x_1, \tau - \gamma h_2) \mathcal{R}_{12}(x_1 - x_2; \tau). \end{aligned}$$

The generator h in (2.9) is an element of the \mathfrak{gl}_2 Cartan subalgebra \mathfrak{h} and \mathcal{L} is regarded as a dynamical operator. Here we are mainly interested on representations of $\mathcal{A}(\mathcal{R})$ consisting of a diagonalizable \mathfrak{h} -module $\widetilde{\mathcal{W}}$ together with a meromorphic function $\mathcal{L}(x, \tau)$ on $\mathbb{C} \times \mathfrak{h}$ with values in $\text{End}_{\mathfrak{h}}(\mathcal{W} \otimes \widetilde{\mathcal{W}})$ such that (2.9) is fulfilled on $\mathcal{W} \otimes \mathcal{W} \otimes \widetilde{\mathcal{W}}$. Therefore, a representation of $\mathcal{A}(\mathcal{R})$ is defined by the pair $(\widetilde{\mathcal{W}}, \mathcal{L})$; and the one consisting of $\widetilde{\mathcal{W}} = \mathcal{W}$ and $\mathcal{L}(x, \tau) = \mathcal{R}(x - \mu; \tau)$, with \mathcal{R} -matrix given by (2.7), is a module over $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$ usually refereed to as fundamental representation with evaluation point μ . For short we also refer to modules over $\mathcal{E}_{p,\gamma}[\widehat{\mathfrak{gl}}_2]$ as \mathcal{E} -modules and remark that more general \mathcal{E} -modules have been constructed in [FV96].

2.4. Dynamical monodromy matrix. One of the building blocks of the Quantum Inverse Scattering Method [STF79, TF79] is the so called *monodromy matrix* and the following theorem paves the way for defining its dynamical version.

Theorem 2.2 (Felder). *Let $(\widetilde{\mathcal{W}}', \mathcal{L}')$ and $(\widetilde{\mathcal{W}}'', \mathcal{L}'')$ be \mathcal{E} -modules. Then $(\widetilde{\mathcal{W}}, \mathcal{L})$ is also an \mathcal{E} -module with $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}' \otimes \widetilde{\mathcal{W}}''$ enjoying \mathfrak{h} -module structure $h(\tilde{w}' \otimes \tilde{w}'') = (h\tilde{w}') \otimes \tilde{w}'' + \tilde{w}' \otimes (h\tilde{w}'')$ for $h \in \mathfrak{h}$, $\tilde{w}' \in \widetilde{\mathcal{W}}'$, $\tilde{w}'' \in \widetilde{\mathcal{W}}''$, and $\mathcal{L} = \mathcal{L}'(x - x', \tau - \gamma h_2) \mathcal{L}''(x - x'', \tau)$.*

Next we would like to build \mathcal{E} -modules, or representations of $\mathcal{A}(\mathcal{R})$, on $\widetilde{\mathcal{W}} = \mathcal{W}^{\otimes n}$. This can be achieved by iterating Theorem 2.2.

Definition 2.3. Let L be a positive integer and fix parameters μ_i for $1 \leq i \leq L$. Also, let $\mathcal{W}_0 = \mathcal{W}_i = \mathcal{W} \simeq \mathbb{C}^2$ and define the dynamical monodromy matrix $\mathcal{T}_0 \in \text{End}(\mathcal{W}_0 \otimes \mathcal{W}^{\otimes L})$ as

$$(2.10) \quad \mathcal{T}_0(x, \tau) := \overrightarrow{\prod}_{1 \leq i \leq L} \mathcal{R}_{0i}(x - \mu_i; \tau - \gamma \sum_{k=i+1}^L h_k) .$$

The object \mathcal{R}_{0i} in (2.10) corresponds to the \mathcal{R} -matrix (2.7) embedded in $\text{End}(\mathcal{W}_0 \otimes \mathcal{W}_i)$. Using Theorem 2.2 one can show the pair $(\mathcal{W}^{\otimes L}, \mathcal{T}_0)$ is an \mathcal{E} -module.

In its turn, $\mathcal{A}(\mathcal{R})$ as an algebra contains relations for the non-commutative entries of $\mathcal{L}(x, \tau)$. Here, however, we shall focus on a particular representation of $\mathcal{A}(\mathcal{R})$, namely $\mathcal{T}_0(x, \tau)$ defined in (2.10). In that case $\mathcal{W} \simeq \mathbb{C}^2$ and the relations in $\mathcal{A}(\mathcal{R})$ can be conveniently described through the structure

$$(2.11) \quad \mathcal{T}_0(x, \tau) =: \begin{pmatrix} \mathcal{A}(x, \tau) & \mathcal{B}(x, \tau) \\ \mathcal{C}(x, \tau) & \mathcal{D}(x, \tau) \end{pmatrix}$$

with generators $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \text{End}(\mathcal{W}^{\otimes L})$.

2.5. Highest-weight modules. One important feature of $\mathcal{A}(\mathcal{R})$ is that the notion of highest-weight module is well defined. The latter was introduced in [FV96] for elliptic quantum groups. The starting point for building highest-weight modules is the concept of *singular vectors* in an \mathcal{E} -module $(\mathcal{W}^{\otimes L}, \mathcal{T}_0)$. They consist of non-zero elements $w_0 \in \mathcal{W}^{\otimes L}$ such that $\mathcal{C}(x, \tau)w_0 = 0$ for all $x, \tau \in \mathbb{C}$. Similarly, we let $\bar{\mathcal{W}}^{\otimes L}$ be the vector space dual to $\mathcal{W}^{\otimes L}$ and call *dual singular vector* the non-zero elements $\bar{w}_0 \in \bar{\mathcal{W}}^{\otimes L}$ such that $\bar{w}_0 \mathcal{C}(x, \tau) = 0$ for all $x, \tau \in \mathbb{C}$.

The representation $\mathcal{W}^{\otimes L}$ is a diagonalizable \mathfrak{h} -module and we say an element $w \in \mathcal{W}^{\otimes L}$ has \mathfrak{h} -weight μ if $hw = \mu w$ for $h \in \mathfrak{h}$. Additionally, we assign the weight $(\mu, \lambda_{\mathcal{A}}(x, \tau), \lambda_{\mathcal{B}}(x, \tau))$ to an element $w \in \mathcal{W}^{\otimes L}$ if $\mathcal{A}(x, \tau)w = \lambda_{\mathcal{A}}(x, \tau)w$, $\mathcal{D}(x, \tau)w = \lambda_{\mathcal{D}}(x, \tau)w$ and w has \mathfrak{h} -weight μ . Also, we restrict our attention to the cases where $\lambda_{\mathcal{A}, \mathcal{D}}$ are non-vanishing meromorphic functions on x and τ .

Now we have gathered the ingredients required to define a highest-weight module. They are formed by singular vectors $w \in \mathcal{W}^{\otimes L}$ having weight $(\mu, \lambda_{\mathcal{A}}(x, \tau), \lambda_{\mathcal{B}}(x, \tau))$. Dual highest-weight modules are defined similarly by considering dual singular vectors instead of singular vectors. Hence, we say $\bar{w} \in \bar{\mathcal{W}}^{\otimes L}$ is a dual highest-weight vector with weight $(\bar{\mu}, \bar{\lambda}_{\mathcal{A}}(x, \tau), \bar{\lambda}_{\mathcal{B}}(x, \tau))$ if it is a dual singular vector satisfying the conditions $\bar{w}h = \bar{\mu}\bar{w}$ for $h \in \mathfrak{h}$, $\bar{w} \mathcal{A}(x, \tau) = \bar{\lambda}_{\mathcal{A}}(x, \tau) \bar{w}$ and $\bar{w} \mathcal{D}(x, \tau) = \bar{\lambda}_{\mathcal{D}}(x, \tau) \bar{w}$.

Next we specialize our discussion to the \mathfrak{gl}_2 case and consider $H := E_{11} - E_{22} \in \mathfrak{h}$. From (2.10) and (2.7) we can take the following vectors as highest- and dual highest-weight vectors respectively:

$$(2.12) \quad w_0 = |0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes L} \quad \text{and} \quad \bar{w}_0 = \langle \bar{0}| := (0 \quad 1)^{\otimes L} .$$

In this way w_0 is a highest-weight vector with weight $(L, \Lambda_{\mathcal{A}}, \Lambda_{\mathcal{D}})$ where

$$(2.13) \quad \begin{aligned} \Lambda_{\mathcal{A}}(x, \tau) &:= \prod_{j=1}^L [x - \mu_j + \gamma] \\ \Lambda_{\mathcal{D}}(x, \tau) &:= \frac{[\tau + \gamma]}{[\tau + (1 - L)\gamma]} \prod_{j=1}^L [x - \mu_j] . \end{aligned}$$

Analogously, \bar{w}_0 is a dual highest-weight vector with weight $(-L, \bar{\Lambda}_{\mathcal{A}}, \bar{\Lambda}_{\mathcal{D}})$ and functions $\bar{\Lambda}_{\mathcal{A}, \mathcal{D}}$ defined as

$$(2.14) \quad \begin{aligned} \bar{\Lambda}_{\mathcal{A}}(x, \tau) &:= \frac{[\tau - \gamma]}{[\tau + (L - 1)\gamma]} \prod_{j=1}^L [x - \mu_j] \\ \bar{\Lambda}_{\mathcal{D}}(x, \tau) &:= \prod_{j=1}^L [x - \mu_j + \gamma] . \end{aligned}$$

2.6. Partition function. The partition function of the $\mathcal{E}_{p, \gamma}[\widehat{\mathfrak{gl}}_2]$ elliptic SOS model with domain-wall boundary conditions admits an operatorial description closely related to that of the six-vertex model [Kor82]. As shown in [Gal13b], it can be written as

$$(2.15) \quad Z_{\tau}(x_1, x_2, \dots, x_L) = \langle \bar{0} | \overrightarrow{\prod}_{1 \leq j \leq L} \mathcal{B}(x_j, \tau + j\gamma) | 0 \rangle$$

with operator \mathcal{B} defined in (2.11). Here we fix the parameters $\gamma, \mu_j \in \mathbb{C}$ and omit their dependence in the LHS of (2.15).

3. ALGEBRAIC-FUNCTIONAL METHOD

The algebra $\mathcal{A}(\mathcal{R})$ associated to the \mathcal{R} -matrix (2.7) is an algebra over \mathbb{C} generated by meromorphic functions f on $h \in \mathfrak{h}$ and elements $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} defined in (2.11). We also refer to $\mathcal{A}(\mathcal{R})$ as *dynamical Yang-Baxter algebra* and it contains two groups of relations. The first group involves generic functions f and g on the Cartan subalgebra \mathfrak{h} with relations reading

$$(3.1) \quad \begin{aligned} f(h)g(h) &= g(h)f(h) \\ \mathcal{A}(x, \tau)f(h) &= f(h)\mathcal{A}(x, \tau) & \mathcal{D}(x, \tau)f(h) &= f(h)\mathcal{D}(x, \tau) \\ \mathcal{B}(x, \tau)f(h) &= f(h+2)\mathcal{B}(x, \tau) & \mathcal{C}(x, \tau)f(h) &= f(h-2)\mathcal{C}(x, \tau) . \end{aligned}$$

The second group is encoded in (2.9).

We also write $\mathcal{A}_2(\mathcal{R})$ for $\mathcal{A}(\mathcal{R})$ regarded as a matrix algebra with elements in $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}] \otimes \text{End}(\mathcal{W}^{\otimes L})$. Moreover, we consider $\mathcal{M}_n := \{\mathcal{A}(x_n, \tau), \mathcal{B}(x_n, \tau), \mathcal{C}(x_n, \tau), \mathcal{D}(x_n, \tau), f(h)\}$ for meromorphic functions f on $h \in \mathfrak{h}$, in such a way that the repeated use of $\mathcal{A}(\mathcal{R})$ yields relations in $\mathcal{A}_n(\mathcal{R}) \cong \mathcal{A}_{n-1}(\mathcal{R}) \otimes \mathcal{M}_n / \mathcal{A}_2(\mathcal{R})$. Here we refer to $\mathcal{A}_n(\mathcal{R})$ as *dynamical Yang-Baxter algebra of degree n* .

The main idea of the AF method is to use $\mathcal{A}_n(\mathcal{R})$ as a source of functional relations characterizing quantities of interest. This is possible if we are able to exhibit a linear functional $\Phi: \mathcal{A}_n(\mathcal{R}) \rightarrow \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ such that

$$(3.2) \quad \begin{aligned} \Phi(J \mathcal{A}(x_n, \tau)) &= \omega_{\mathcal{A}}(x_n, \tau) \Phi(J) & \Phi(J \mathcal{D}(x_n, \tau)) &= \omega_{\mathcal{D}}(x_n, \tau) \Phi(J) \\ \Phi(\mathcal{A}(x_n, \tau) J) &= \bar{\omega}_{\mathcal{A}}(x_n, \tau) \Phi(J) & \Phi(\mathcal{D}(x_n, \tau) J) &= \bar{\omega}_{\mathcal{D}}(x_n, \tau) \Phi(J) \end{aligned}$$

for fixed functions $\omega_{\mathcal{A}, \mathcal{D}}$, $\bar{\omega}_{\mathcal{A}, \mathcal{D}}$ and $J \in \mathcal{A}_{n-1}(\mathcal{R})$. The construction of Φ will then depend on the quantity we would like to describe.

Proposition 3.1. *Taking into account the operatorial description (2.15) we can conveniently realize Φ as*

$$(3.3) \quad \Phi(J) = \langle \bar{0} | J | 0 \rangle \quad J \in \mathcal{A}_n(\mathcal{R}),$$

with vectors $\langle \bar{0} |$ and $|0\rangle$ defined in (2.12).

Proof. Direct use of properties described in Section 2.5. □

Remark 3.2. Conditions (3.2) are fulfilled with $\omega_{\mathcal{A}, \mathcal{D}} = \Lambda_{\mathcal{A}, \mathcal{D}}$ and $\bar{\omega}_{\mathcal{A}, \mathcal{D}} = \bar{\Lambda}_{\mathcal{A}, \mathcal{D}}$ defined in (2.13) and (2.14).

In what follows we shall use the above described procedure to derive two functional relations, namely equations type A and type D, characterizing the partition function of the elliptic SOS model with domain-wall boundaries.

3.1. Equation type A. The algebra $\mathcal{A}_2(\mathcal{R})$ amounts to 21 commutation relations: 5 involving functions on \mathfrak{h} (3.1) and 16 encoded in (2.9). In order to obtain a functional relation of *type A*, we can restrict ourselves to the subalgebra $\mathcal{S}_{\mathcal{A}, \mathcal{B}} \subset \mathcal{A}_2(\mathcal{R})$ formed by

$$(3.4) \quad \begin{aligned} \mathcal{A}(x_1, \tau) \mathcal{A}(x_2, \tau - \gamma) &= \mathcal{A}(x_2, \tau) \mathcal{A}(x_1, \tau - \gamma) \\ \mathcal{B}(x_1, \tau) \mathcal{B}(x_2, \tau + \gamma) &= \mathcal{B}(x_2, \tau) \mathcal{B}(x_1, \tau + \gamma) \\ \mathcal{A}(x_1, \tau) \mathcal{B}(x_2, \tau - \gamma) &= \frac{[x_2 - x_1 + \gamma][\tau]}{[x_2 - x_1][\tau + \gamma]} \mathcal{B}(x_2, \tau) \mathcal{A}(x_1, \tau + \gamma) \\ &\quad + \frac{[\tau + x_1 - x_2][\gamma]}{[x_1 - x_2][\tau + \gamma]} \mathcal{B}(x_1, \tau) \mathcal{A}(x_2, \tau + \gamma) \\ \mathcal{B}(x_1, \tau) \mathcal{A}(x_2, \tau + \gamma) &= \frac{[x_2 - x_1 + \gamma][\tau]}{[x_2 - x_1][\tau - \gamma]} \mathcal{A}(x_2, \tau) \mathcal{B}(x_1, \tau - \gamma) \\ &\quad - \frac{[\tau + x_2 - x_1][\gamma]}{[x_2 - x_1][\tau - \gamma]} \mathcal{A}(x_1, \tau) \mathcal{B}(x_2, \tau - \gamma). \end{aligned}$$

In particular, the iteration of $\mathcal{S}_{\mathcal{A},\mathcal{B}}$ yields the following relation in $\mathcal{A}_{L+1}(\mathcal{R})$,

$$\begin{aligned}
 & \mathcal{A}(x_0, \tau + 2\gamma) Y_\tau(X) = \\
 & \frac{[\tau + 2\gamma]}{[\tau + (L + 2)\gamma]} \prod_{j=1}^L \frac{[x_j - x_0 + \gamma]}{[x_j - x_0]} Y_{\tau+\gamma}(X) \mathcal{A}(x_0, \tau + (L + 2)\gamma) \\
 & + \sum_{i=1}^L \frac{[\tau + 2\gamma + x_0 - x_i][\gamma]}{[x_0 - x_i][\tau + (L + 2)\gamma]} \prod_{\substack{j=1 \\ j \neq i}}^L \frac{[x_j - x_i + \gamma]}{[x_j - x_i]} Y_{\tau+\gamma}(X_i^0) \mathcal{A}(x_i, \tau + (L + 2)\gamma) ,
 \end{aligned}
 \tag{3.5}$$

where $X := \{x_1, x_2, \dots, x_L\}$, $X_i^\alpha := X \cup \{x_\alpha\} \setminus \{x_i\}$ and $Y_\tau(X) := \prod_{1 \leq j \leq L} \mathcal{B}(x_j, \tau + j\gamma)$.

Theorem 3.3 (Equation type A). *The partition function Z_τ satisfies the functional equation*

$$(3.6) \quad M_0^{(\mathcal{A})} Z_\tau(X) + \sum_{i \in \{0, 1, 2, \dots, L\}} N_i^{(\mathcal{A})} Z_{\tau+\gamma}(X_i^0) = 0 ,$$

with coefficients given by

$$\begin{aligned}
 M_0^{(\mathcal{A})} &:= \frac{[\tau + \gamma]}{[\tau + (L + 1)\gamma]} \prod_{j=1}^L [x_0 - \mu_j] \\
 N_0^{(\mathcal{A})} &:= -\frac{[\tau + 2\gamma]}{[\tau + (L + 2)\gamma]} \prod_{j=1}^L [x_0 - \mu_j + \gamma] \prod_{j=1}^L \frac{[x_j - x_0 + \gamma]}{[x_j - x_0]} \\
 N_i^{(\mathcal{A})} &:= \frac{[\tau + 2\gamma + x_0 - x_i][\tau]}{[x_i - x_0][\tau + (L + 2)\gamma]} \prod_{j=1}^L [x_i - \mu_j + \gamma] \prod_{\substack{j=1 \\ j \neq i}}^L \frac{[x_j - x_i + \gamma]}{[x_j - x_i]} \quad i = 1, 2, \dots, L .
 \end{aligned}
 \tag{3.7}$$

Proof. We firstly remark that $Y_\tau(X) \in \mathcal{A}_L(\mathcal{R})$ and apply the functional Φ given by (3.3) onto (3.5). Then we only need to identify $\Phi(Y_\tau(X)) = Z_\tau(X)$ using (2.15). \square

Remark 3.4. Eq. (3.6) is the very same functional equation derived in [Gal13b]. The precise matching can be achieved by performing a trivial shift in the dynamical parameter.

3.2. Equation type D. Formula (3.5) is not the only potentially useful relation in $\mathcal{A}_{L+1}(\mathcal{R})$. In fact, one can find a large number of relations in $\mathcal{A}_n(\mathcal{R})$ ($n \geq L + 1$) which can be exploited along the lines of the AF framework in order to describe the partition function (2.15). However, the simplest ones considered so far seem to have structure resembling (3.6). One of the goals of the present paper is to obtain determinantal representations for Z_τ and that will also require an equation of *type D* analogous to (3.6). The derivation of

such equation within the AF framework will then exploit the subalgebra $\mathcal{S}_{\mathcal{D},\mathcal{B},H} \subset \mathcal{A}_2(\mathcal{R})$ formed by

$$\begin{aligned} \mathcal{D}(x_1, \tau) f(H) &= f(H) \mathcal{D}(x_1, \tau) & \mathcal{D}(x_1, \tau) \mathcal{D}(x_2, \tau + \gamma) &= \mathcal{D}(x_2, \tau) \mathcal{D}(x_1, \tau + \gamma) \\ \mathcal{B}(x_1, \tau) f(H) &= f(H + 2) \mathcal{B}(x_1, \tau) & \mathcal{B}(x_1, \tau) \mathcal{B}(x_2, \tau + \gamma) &= \mathcal{B}(x_2, \tau) \mathcal{B}(x_1, \tau + \gamma) \end{aligned}$$

$$\begin{aligned} [\tau + \gamma(1 - H)] \mathcal{D}(x_1, \tau) \mathcal{B}(x_2, \tau + \gamma) &= \\ \frac{[x_1 - x_2 + \gamma]}{[x_1 - x_2]} [\tau - \gamma H] \mathcal{B}(x_2, \tau) \mathcal{D}(x_1, \tau + \gamma) & \\ - \frac{[\gamma]}{[x_1 - x_2]} [\tau + x_1 - x_2 - \gamma H] \mathcal{B}(x_1, \tau) \mathcal{D}(x_2, \tau + \gamma) & \\ [\tau - \gamma(1 + H)] \mathcal{B}(x_1, \tau) \mathcal{D}(x_2, \tau + \gamma) &= \\ \frac{[x_1 - x_2 + \gamma]}{[x_1 - x_2]} [\tau - \gamma H] \mathcal{D}(x_2, \tau) \mathcal{B}(x_1, \tau + \gamma) & \\ - \frac{[\gamma]}{[x_1 - x_2]} [\tau + x_2 - x_1 - \gamma H] \mathcal{D}(x_1, \tau) \mathcal{B}(x_2, \tau + \gamma) . & \end{aligned}$$

(3.8)

The subalgebra $\mathcal{S}_{\mathcal{D},\mathcal{B},H}$ is generated by \mathcal{D} , \mathcal{B} and the Cartan element H ; and in order to proceed we also need an analogous of (3.5) associated to the iteration of $\mathcal{S}_{\mathcal{D},\mathcal{B},H}$. Similarly to derivation presented in Section 3.1, the iteration of $\mathcal{S}_{\mathcal{D},\mathcal{B},H}$ produces the following relation in $\mathcal{A}_{L+1}(\mathcal{R})$,

$$\begin{aligned} \mathcal{D}(x_{\bar{0}}, \tau + \gamma) Y_{\tau+\gamma}(X) &= \\ [\tau + 2\gamma] \prod_{j=1}^L \frac{[x_{\bar{0}} - x_j + \gamma]}{[x_{\bar{0}} - x_j]} Y_{\tau}(X) \mathcal{D}(x_{\bar{0}}, \tau + \gamma(L + 1)) [\tau + \gamma(H + 2)]^{-1} & \\ - \sum_{i=1}^L \frac{[\gamma][\tau + 2\gamma]}{[x_{\bar{0}} - x_i]} \prod_{\substack{j=1 \\ j \neq i}}^L \frac{[x_i - x_j + \gamma]}{[x_i - x_j]} Y_{\tau}(X_i^{\bar{0}}) \mathcal{D}(x_i, \tau + \gamma(L + 1)) [\tau + \gamma(H + 1) + x_{\bar{0}} - x_i] & \\ \times [\tau + \gamma(H + 1)]^{-1} [\tau + \gamma(H + 2)]^{-1} . & \end{aligned}$$

(3.9)

Theorem 3.5 (Equation type D). *The functional relation*

$$(3.10) \quad M_0^{(\mathcal{D})} Z_{\tau+\gamma}(X) + \sum_{i \in \{\bar{0}, 1, 2, \dots, L\}} N_i^{(\mathcal{D})} Z_{\tau}(X_i^{\bar{0}}) = 0 ,$$

with coefficients reading

$$\begin{aligned}
M_0^{(\mathcal{D})} &:= \prod_{j=1}^L [x_{\bar{0}} - \mu_j + \gamma] \\
N_0^{(\mathcal{D})} &:= - \prod_{j=1}^L [x_{\bar{0}} - \mu_j] \prod_{j=1}^L \frac{[x_{\bar{0}} - x_j + \gamma]}{[x_{\bar{0}} - x_j]} \\
N_i^{(\mathcal{D})} &:= \frac{[\gamma][\tau + (L+1)\gamma + x_{\bar{0}} - x_i]}{[x_{\bar{0}} - x_i][\tau + (L+1)\gamma]} \prod_{j=1}^L [x_i - \mu_j] \prod_{\substack{j=1 \\ j \neq i}}^L \frac{[x_i - x_j + \gamma]}{[x_i - x_j]} \quad i = 1, 2, \dots, L,
\end{aligned}
\tag{3.11}$$

is satisfied by the partition function (2.15).

Proof. Same as for Theorem 3.3 but using (3.9) instead of (3.5). \square

Remark 3.6. The terminology *Equations type A and D*, respectively associated to (3.6) and (3.10), are intrinsically associated to their roots in relations (3.5) and (3.9).

4. FUNCTIONAL RELATION AD

In [Gal13b] we have shown how Eq. type A can be solved in terms of a multiple contour integral. A similar treatment can be performed on Eq. type D yielding a comparable representation. Although each equation can be solved separately using the procedure described in [Gal13b], here we shall adopt a different approach for the resolution of (3.6) and (3.10). Our goal here is to find determinantal representations for the partition function Z_τ and we anticipate such representations emerges naturally from a particular combination of Eqs. type A and D. In order to proceed it is convenient to introduce the following definition.

Definition 4.1 (Permutation). *Let $n \in \mathbb{Z}_{>0}$ and \mathcal{S}_n be the symmetric group in n letters acting by permutations on \mathbb{C}^n . Also, for $1 \leq i, j \leq n$ write $\Pi_{i,j} \in \mathcal{S}_n$ for the permutation of letters i and j ; and let $\text{Fun}(\mathbb{C}^n)$ be the space of meromorphic functions on \mathbb{C}^n . The symmetric group \mathcal{S}_n then acts on $f \in \text{Fun}(\mathbb{C}^n)$ by*

$$(\Pi_{i,j} f)(x_1, \dots, x_i, \dots, x_j, \dots, x_n) := f(x_1, \dots, x_j, \dots, x_i, \dots, x_n) .$$

Theorem 4.2 (Equation type AD). *The partition function (2.15) fulfills the relation*

$$\mathcal{M}_0 Z_\tau(X) + \sum_{i=1}^L \mathcal{N}_i Z_\tau(X_i^0) + \sum_{i=1}^L \bar{\mathcal{N}}_i Z_\tau(X_i^{\bar{0}}) = 0 ,
\tag{4.1}$$

with coefficients defined as

$$\begin{aligned}
\mathcal{M}_0 &:= \prod_{j=1}^L \frac{[x_0 - x_j + \gamma][x_0 - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_0 - x_j][x_0 - \mu_j + \gamma]} - \prod_{j=1}^L \frac{[x_{\bar{0}} - x_j + \gamma][x_{\bar{0}} - \mu_j]}{[x_{\bar{0}} - x_j]} \\
\mathcal{N}_i &:= -\frac{[\gamma][x_0 - x_i + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_0 - x_i]} \prod_{j=1}^L \frac{[x_i - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \prod_{\substack{j=1 \\ j \neq i}}^L \frac{[x_i - x_j + \gamma]}{[x_i - x_j]} \\
\bar{\mathcal{N}}_i &:= \frac{[\gamma][x_{\bar{0}} - x_i + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_{\bar{0}} - x_i]} \prod_{j=1}^L [x_i - \mu_j] \prod_{\substack{j=1 \\ j \neq i}}^L \frac{[x_i - x_j + \gamma]}{[x_i - x_j]}.
\end{aligned}
\tag{4.2}$$

Proof. We start with a careful analysis of (3.6) under the action of $\Pi_{0,l}$ for $0 \leq l \leq L$. The case $l = 0$ is regarded as the original equation. These permutations result in the same type of functional equation but with modified coefficients. Hence (3.6) actually consists of a system of linear equations and we remark this property is a consequence of Z_τ being a symmetric function. In fact, we have shown in [Gal13b] that (3.6) only admits analytic solutions which are symmetric. In this way the action of $\Pi_{0,l}$ on (3.6) results in a set of $L + 1$ equations, namely

$$(4.3) \quad M_0^{(\mathcal{A},l)} Z_\tau(X_l^0) + \sum_{i=0}^L N_i^{(\mathcal{A},l)} Z_{\tau+\gamma}(X_i^0) = 0,$$

with coefficients

$$M_0^{(\mathcal{A},l)} := \Pi_{0,l} M_0^{(\mathcal{A})} \quad \text{and} \quad N_i^{(\mathcal{A},l)} := \begin{cases} \Pi_{0,l} N_l^{(\mathcal{A})} & i = 0 \\ \Pi_{0,l} N_0^{(\mathcal{A})} & i = l \\ \Pi_{0,l} N_i^{(\mathcal{A})} & \text{otherwise} \end{cases}.$$

Next we notice that each equation (4.3) contains $L + 1$ terms of the form $Z_{\tau+\gamma}(X_i^0)$. Therefore, we can solve our system of $L + 1$ equations for each one of those terms. This procedure allows us, for instance, to express $Z_{\tau+\gamma}(X)$ as a linear combination of terms $Z_\tau(X_l^0)$ with index l taking values on the set $\{0, 1, \dots, L\}$. By doing so, with the help of Cramer's method and Laplace expansion, we find

$$(4.4) \quad Z_{\tau+\gamma}(X) = \sum_{l=0}^L (-1)^{l+1} M_0^{(\mathcal{A},l)} \frac{\det \left(N_j^{(\mathcal{A},i)} \right)_{\substack{0 \leq i \leq L, i \neq l \\ 1 \leq j \leq L}}}{\det \left(N_j^{(\mathcal{A},i)} \right)_{0 \leq i, j \leq L}} Z_\tau(X_l^0).$$

Formula (4.4) can be straightforwardly substituted into (3.10) and after simplifications we are left with (4.1). This completes our proof. \square

Now we are ready to discuss the implications of Theorem 4.2. When comparing (3.6) and (3.10) with equation (4.1) one can readily notice that the dynamical parameter τ no

longer plays the role of variable. All terms in (4.1) are evaluated at the same value of τ and it can be fixed from this point on. On the other hand, the structure of (4.1) is more complicated than that of Eqs. (3.6) and (3.10) as it now depends on two extra variables, namely x_0 and $x_{\bar{0}}$, in addition to the set $\{x_1, x_2, \dots, x_L\}$ required to describe the partition function Z_τ . However, we shall see that these additional variables play an important role for solving (4.1) in terms of a single determinant.

Remark 4.3. The mechanism leading to (4.1) considers Eq. type A (3.6) under permutation of variables and replaces the result into Eq. type D (3.10). On the other hand, one could have used permutations of (3.10) and substituted the result into (3.6). The latter procedure yields an equation comparable to (4.1) but now in terms of functions $Z_{\tau+\gamma}$. Unexpectedly, these equations are not related by a shift in the dynamical parameter τ and they actually seem to be independent.

5. PARTITION FUNCTION EVALUATION

The resolution of Eq. (4.1) is the main purpose of this section. For that we shall employ a generalization of the method recently put forward in [Gal16b] which allows solutions of this type of equations to be obtained in terms of determinants. Although both Eqs. (3.6) and (3.10) are already able to fix the partition function Z_τ , their dependence with the dynamical parameter τ seems to prevent the direct use of the methodology described in [Gal16b]. This is the main reason for considering (4.1) instead of (3.6) or (3.10). Eq. (4.1) downgrades the status of the dynamical parameter τ but this feature does not come for free. The price we pay is the introduction of one extra spectral variable in addition to the $L+1$ already existing in (3.6) and (3.10). This additional variable makes the resolution of (4.1) a bit more involving.

We start our analysis by remarking some features of Eq. (4.1) originated from the distinct role played by the variables x_0 and $x_{\bar{0}}$. For instance, at first sight it seems that (4.1) reduces to the same type of functional equation obtained for the six-vertex model in [Gal13a, Gal13b] under the specialization $x_0 = x_{\bar{0}}$. However, this is not the case and one can readily see that Eq. (4.1) is automatically satisfied when this particular condition is fulfilled. Hence, we eliminate this point from our analysis.

Next we study the behavior of equation type AD under permutations of variables. The first important observation is that (4.1) is invariant under the action of $\Pi_{i,j}$ for $1 \leq i, j \leq L$. On the other hand, permutations $\Pi_{0,l}$ and $\Pi_{\bar{0},m}$ for $0 \leq l, m \leq L$ produces new equations whose structure escapes the one given by (4.1). Nevertheless, those new equations can be accommodated in

$$\mathcal{M}_0^{(l,m)} Z_\tau(X) + \sum_{i=1}^L \mathcal{N}_i^{(l,m)} Z_\tau(X_i^0) + \sum_{i=1}^L \bar{\mathcal{N}}_i^{(l,m)} Z_\tau(X_i^{\bar{0}}) + \sum_{1 \leq i,j \leq L} \mathcal{O}_{ij}^{(l,m)} Z_\tau(X_{i,j}^{0,\bar{0}}) = 0, \quad (5.1)$$

where we have introduced the additional notation $X_{i,j}^{\alpha,\beta} := X \cup \{x_\alpha, x_\beta\} \setminus \{x_i, x_j\}$. The label (l, m) in the coefficients of (5.1) also requires some clarifications. The coefficients with label

$(l, m) = (l, \bar{0})$ arise from (4.1) under permutation $\Pi_{0,l}$ while keeping $x_{\bar{0}}$ fixed. Similarly, Eq. (5.1) with label $(l, m) = (0, m)$ results from (4.1) under the action of $\Pi_{\bar{0},m}$ while x_0 is now kept fixed. As for the remaining cases; that is (5.1) with label (l, m) restricted to $1 \leq l < m \leq L$, they are obtained through permutations $\Pi_{0,l}$ and $\Pi_{\bar{0},m}$ in this respective order. Taking into account the aforementioned permutation of variables, we are then left with the following expressions for the coefficients in (5.1):

$$(5.2) \quad \begin{aligned} \mathcal{M}_0^{(l,\bar{0})} &:= \Pi_{0,l} \mathcal{N}_l & \bar{\mathcal{N}}_j^{(l,\bar{0})} &:= \begin{cases} \Pi_{0,l} \bar{\mathcal{N}}_l & j = l \\ 0 & \text{otherwise} \end{cases} \\ \mathcal{N}_j^{(l,\bar{0})} &:= \begin{cases} \Pi_{0,l} \mathcal{M}_0 & j = l \\ \Pi_{0,l} \mathcal{N}_j & \text{otherwise} \end{cases} & \mathcal{O}_{ij}^{(l,\bar{0})} &:= \begin{cases} \Pi_{0,l} \bar{\mathcal{N}}_i & j = l \\ \Pi_{0,l} \bar{\mathcal{N}}_j & i = l \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$(5.3) \quad \begin{aligned} \mathcal{M}_0^{(0,m)} &:= \Pi_{\bar{0},m} \bar{\mathcal{N}}_m & \bar{\mathcal{N}}_j^{(0,m)} &:= \begin{cases} \Pi_{\bar{0},m} \mathcal{M}_0 & j = m \\ \Pi_{\bar{0},m} \bar{\mathcal{N}}_j & \text{otherwise} \end{cases} \\ \mathcal{N}_j^{(0,m)} &:= \begin{cases} \Pi_{\bar{0},m} \mathcal{N}_m & j = m \\ 0 & \text{otherwise} \end{cases} & \mathcal{O}_{ij}^{(0,m)} &:= \begin{cases} \Pi_{\bar{0},m} \mathcal{N}_i & j = m \\ \Pi_{\bar{0},m} \mathcal{N}_j & i = m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$(5.4) \quad \begin{aligned} \mathcal{M}_0^{(l,m)} &:= 0 & \bar{\mathcal{N}}_j^{(l,m)} &:= \begin{cases} \Pi_{\bar{0},m} \circ \Pi_{0,l} \bar{\mathcal{N}}_l & j = l \\ \Pi_{\bar{0},m} \circ \Pi_{0,l} \bar{\mathcal{N}}_l & j = m \\ 0 & \text{otherwise} \end{cases} \\ \mathcal{N}_j^{(l,m)} &:= \begin{cases} \Pi_{\bar{0},m} \circ \Pi_{0,l} \bar{\mathcal{N}}_m & j = l \\ \Pi_{\bar{0},m} \circ \Pi_{0,l} \mathcal{N}_m & j = m \\ 0 & \text{otherwise} \end{cases} & \mathcal{O}_{ij}^{(l,m)} &:= \begin{cases} \Pi_{\bar{0},m} \circ \Pi_{0,l} \mathcal{M}_0 & i = l, j = m \\ \Pi_{\bar{0},m} \circ \Pi_{0,l} \bar{\mathcal{N}}_j & i = l, j \neq m \\ \Pi_{\bar{0},m} \circ \Pi_{0,l} \mathcal{N}_j & i = m \\ \Pi_{\bar{0},m} \circ \Pi_{0,l} \bar{\mathcal{N}}_i & j = l \\ \Pi_{\bar{0},m} \circ \Pi_{0,l} \mathcal{N}_i & j = m, i \neq l \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Remark 5.1. The structure of (5.1) resembles that of the equation presented in [Gal12] for the trigonometric SOS model with domain-wall boundaries. However, there are still crucial differences and the existence of a precise relation is not clear at the moment.

In order to proceed let us have a closer look into the system of equations (5.1). One can clearly see this system comprises $d_L := L(L+3)/2$ equations: L equations coming from permutations $\Pi_{0,l}$, another L equations originated from the action of $\Pi_{0,m}$ and $L(L-1)/2$ obtained from the composed action $\Pi_{0,m} \circ \Pi_{0,l}$ under the restriction $1 \leq l < m \leq L$. In complement to that we can see the system of Eqs. (5.1) encloses $d_L + 1$ unknowns. They are the L terms of the form $Z_\tau(X_i^0)$, another L of type $Z_\tau(X_i^{\bar{0}})$, $L(L-1)/2$ terms recasted as $Z_\tau(X_{i,j}^{0,\bar{0}})$ and one term $Z_\tau(X)$. Hence, for instance, we have enough equations to express each function $Z_\tau(X_i^0)$, $Z_\tau(X_i^{\bar{0}})$ and $Z_\tau(X_{i,j}^{0,\bar{0}})$ solely in terms of $Z_\tau(X)$. This can be algorithmically implemented using Cramer's method. By doing so we find

$$(5.5) \quad Z_\tau(X_i^0) = \frac{\begin{vmatrix} \mathcal{F}_i & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}}_i & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}}_i & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{vmatrix}}{\begin{vmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{vmatrix}} Z_\tau(X) \quad , \quad Z_\tau(X_i^{\bar{0}}) = \frac{\begin{vmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G}_i \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J}_i \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}}_i \end{vmatrix}}{\begin{vmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{vmatrix}} Z_\tau(X)$$

and

$$(5.6) \quad Z_\tau(X_{i,j}^{0,\bar{0}}) = \frac{\begin{vmatrix} \mathcal{F} & \mathcal{I}_{ij} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K}_{ij} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}}_{ij} & \bar{\mathcal{G}} \end{vmatrix}}{\begin{vmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{vmatrix}} Z_\tau(X) .$$

The matrix coefficients entering the determinants in (5.5) and (5.6) follow directly from Eq. (5.1). More precisely, the coefficients \mathcal{F} , $\bar{\mathcal{F}}$, \mathcal{G} and $\bar{\mathcal{G}}$ are submatrices of dimension $L \times L$ with entries defined as

$$(5.7) \quad \begin{aligned} \mathcal{F}_{a,b} &:= \mathcal{N}_b^{(a,\bar{0})} & \mathcal{G}_{a,b} &:= \bar{\mathcal{N}}_b^{(a,\bar{0})} \\ \bar{\mathcal{F}}_{a,b} &:= \mathcal{N}_b^{(0,a)} & \bar{\mathcal{G}}_{a,b} &:= \bar{\mathcal{N}}_b^{(0,a)} . \end{aligned}$$

As for the remaining submatrices, it is convenient to introduce an index $n: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defined explicitly as $n_{r,s} := s + L(r-1) - \frac{r(r+1)}{2}$ in the domain $1 \leq r < s \leq L$. In this way, the matrices \mathcal{I} and $\bar{\mathcal{J}}$ are of dimension $L \times \frac{L(L-1)}{2}$ with entries defined as

$$(5.8) \quad \mathcal{I}_{a,n_{r,s}} := \mathcal{O}_{rs}^{(a,\bar{0})} \quad \text{and} \quad \bar{\mathcal{J}}_{a,n_{r,s}} := \mathcal{O}_{rs}^{(0,a)} .$$

Next we turn our attention to the matrices $\bar{\mathcal{I}}$ and \mathcal{J} . Their entries are defined as

$$(5.9) \quad \bar{\mathcal{I}}_{n_{l,m},b} := \mathcal{N}_b^{(l,m)} \quad \text{and} \quad \mathcal{J}_{n_{l,m},b} := \bar{\mathcal{N}}_b^{(l,m)} .$$

The latter definition builds up matrices of dimension $\frac{L(L-1)}{2} \times L$. In its turn the matrix \mathcal{K} has dimension $\frac{L(L-1)}{2} \times \frac{L(L-1)}{2}$ and its entries are defined as

$$(5.10) \quad \mathcal{K}_{n_{l,m},n_{r,s}} := \mathcal{O}_{rs}^{(l,m)} .$$

Here we recall that $1 \leq l < m \leq L$ and $1 \leq r < s \leq L$.

Definitions (5.7)-(5.10) allow one to write down only the matrix appearing in the denominator of relations (5.5) and (5.6). As for the remaining matrices appearing in the numerators we still need to define the tuples of matrices $(\mathcal{F}_i, \bar{\mathcal{I}}_i, \bar{\mathcal{F}}_i)$, $(\mathcal{G}_i, \mathcal{J}_i, \bar{\mathcal{G}}_i)$ and $(\mathcal{I}_{ij}, \mathcal{K}_{ij}, \bar{\mathcal{J}}_{ij})$. Expressions (5.5) and (5.6) are obtained from Cramer's rule and consequently the remaining matrices do not differ drastically from the ones already presented in (5.7)-(5.10). In this way the elements of the first tuple $(\mathcal{F}_i, \bar{\mathcal{I}}_i, \bar{\mathcal{F}}_i)$ have entries

$$(\mathcal{F}_i)_{a,b} := \begin{cases} -\mathcal{M}_0^{(a,\bar{0})} & i = b \\ \mathcal{N}_b^{(a,\bar{0})} & \text{otherwise} \end{cases}, \quad (\bar{\mathcal{I}}_i)_{n_{l,m},b} := \begin{cases} -\mathcal{M}_0^{(l,m)} & i = b \\ \mathcal{N}_b^{(l,m)} & \text{otherwise} \end{cases} \quad (5.11)$$

and

$$(\bar{\mathcal{F}}_i)_{a,b} := \begin{cases} -\mathcal{M}_0^{(0,a)} & i = b \\ \mathcal{N}_b^{(0,a)} & \text{otherwise} \end{cases}. \quad (5.12)$$

As for the tuple $(\mathcal{G}_i, \mathcal{J}_i, \bar{\mathcal{G}}_i)$ we have

$$(\mathcal{G}_i)_{a,b} := \begin{cases} -\mathcal{M}_0^{(a,\bar{0})} & i = b \\ \bar{\mathcal{N}}_b^{(a,\bar{0})} & \text{otherwise} \end{cases}, \quad (\mathcal{J}_i)_{n_{l,m},b} := \begin{cases} -\mathcal{M}_0^{(l,m)} & i = b \\ \bar{\mathcal{N}}_b^{(l,m)} & \text{otherwise} \end{cases} \quad (5.13)$$

and

$$(\bar{\mathcal{G}}_i)_{a,b} := \begin{cases} -\mathcal{M}_0^{(0,a)} & i = b \\ \bar{\mathcal{N}}_b^{(0,a)} & \text{otherwise} \end{cases}. \quad (5.14)$$

Lastly, the elements of the tuple $(\mathcal{I}_{ij}, \mathcal{K}_{ij}, \bar{\mathcal{J}}_{ij})$ are built with the help of the following definitions:

$$\begin{aligned} (\mathcal{I}_{ij})_{a,n_{r,s}} &:= \begin{cases} -\mathcal{M}_0^{(a,\bar{0})} & i = r, j = s \\ \mathcal{O}_{rs}^{(a,\bar{0})} & \text{otherwise} \end{cases} \\ (\mathcal{K}_{ij})_{n_{l,m},n_{r,s}} &:= \begin{cases} -\mathcal{M}_0^{(l,m)} & i = r, j = s \\ \mathcal{O}_{rs}^{(l,m)} & \text{otherwise} \end{cases} \\ (\bar{\mathcal{J}}_{ij})_{a,n_{r,s}} &:= \begin{cases} -\mathcal{M}_0^{(0,a)} & i = r, j = s \\ \mathcal{O}_{rs}^{(0,a)} & \text{otherwise} \end{cases}. \end{aligned} \quad (5.15)$$

In (5.11)-(5.15) we stress again that our relations are defined in the domains $1 \leq l < m \leq L$ and $1 \leq r < s \leq L$. Also, from (5.2)-(5.4) one can notice some coefficients in Eq. (5.1)

vanish and consequently the determinants appearing in (5.5) and (5.6) are not taken over full matrices.

5.1. Determinantal representations. The analysis of equation type AD (4.1) has led us to the system of equations (5.1), from which (5.5) and (5.6) follows as a consequence. In particular, the structure of (5.5) and (5.6) is quite appealing from the perspective of expressing Z_τ as a determinant. The latter is our goal in this section and for that we firstly take a closer look at the structure of (5.5) and (5.6). For instance, the first equation in (5.5) says $Z_\tau(X_i^0) \sim Z_\tau(X)$ and for such kind of relation we can fix all variables in the set $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_L\}$. The same argument also holds for the second relation in (5.5). Thus each equation in (5.5) is essentially an one-variable functional relation which could be simply regarded as $Z_\tau(x_0) \sim Z_\tau(x_i)$ and $Z_\tau(x_{\bar{0}}) \sim Z_\tau(x_i)$. Similarly, Eq. (5.6) can be regarded as a two-variable functional relation, i.e. $Z_\tau(x_0, x_{\bar{0}}) \sim Z_\tau(x_i, x_j)$. This analysis paves the way for using *separation of variables* for solving the aforementioned functional relations.

The fact that (5.5) and (5.6) hold has important consequences from the perspective of separation of variables. Those relations tell us that the ratio of determinants defined by (5.11)-(5.15) necessarily simplifies unveiling the solution up to an overall multiplicative factor. More precisely, from (5.5) and (5.6) we can infer that

$$\begin{aligned}
 \begin{vmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{vmatrix} &= Z_\tau(X) f(x_0, x_{\bar{0}}, x_1, \dots, x_L) & \begin{vmatrix} \mathcal{F}_i & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}}_i & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}}_i & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{vmatrix} &= Z_\tau(X_i^0) f(x_0, x_{\bar{0}}, x_1, \dots, x_L) \\
 \begin{vmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G}_i \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J}_i \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}}_i \end{vmatrix} &= Z_\tau(X_i^{\bar{0}}) f(x_0, x_{\bar{0}}, x_1, \dots, x_L) & \begin{vmatrix} \mathcal{F} & \mathcal{I}_{ij} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K}_{ij} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}}_{ij} & \bar{\mathcal{G}} \end{vmatrix} &= Z_\tau(X_{i,j}^{0,\bar{0}}) f(x_0, x_{\bar{0}}, x_1, \dots, x_L)
 \end{aligned}
 \tag{5.16}$$

for a given fixed function f . Thus our problem actually consists of extracting the partition function Z_τ out of those determinants.

Initially let us focus on the first relation of (5.5) which expresses $Z_\tau(X_i^0)/Z_\tau(X)$ as the ratio of two determinants. From the inspection of (5.7)-(5.12), (5.2)-(5.4) and (4.2) one can verify the matrix coefficients entering those determinants depend on the set of variables X as well as x_0 and $x_{\bar{0}}$. However, $Z_\tau(X_i^0)/Z_\tau(X)$ is independent of $x_{\bar{0}}$ and consequently the corresponding ratio of determinants inherits this property. In this way, although the matrix entries (5.7)-(5.12) exhibit local dependence on $x_{\bar{0}}$, this variable has no global effect as far as the first relation of (5.5) is concerned. The second relation of (5.5) and (5.6) allow one to draw similar conclusions involving the variable x_0 . We shall postpone the discussion on the role played by x_0 and $x_{\bar{0}}$; and proceed with the examination of the matrix

$$\Omega := \begin{pmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{pmatrix}.
 \tag{5.17}$$

We want to extract $Z_\tau(X)$ from $\det(\Omega)$ and, taking into account (5.2)-(5.4) and (5.7)-(5.10), we can see \mathcal{F} and \mathcal{G} are full matrices whose diagonal entries consist of a sum of two products and off-diagonal entries given by a single product. In their turn $\bar{\mathcal{F}}$ and \mathcal{G} are diagonal matrices with non-vanishing components given by single products. The matrices \mathcal{I} , $\bar{\mathcal{I}}$, \mathcal{J} and $\bar{\mathcal{J}}$ are sparse and their non-null entries are single products. Lastly, \mathcal{K} is also sparse but with diagonal components given by the sum of two products and off-diagonal ones consisting of single products. The analysis of relations (5.16) then allow us to state the following theorem.

Theorem 5.2. *The partition function $Z_\tau(X)$ can be written as*

$$(5.18) \quad \begin{aligned} Z_\tau(X) = & (-1)^L \left(\frac{[(L+1)\gamma]}{[\tau + (L+2)\gamma]} \right)^{d_L-1} \left(\frac{[\tau + (L+1)\gamma]}{[L\gamma]} \right)^{d_L} \prod_{i,j=1}^L [x_i - \mu_j] \prod_{k=1}^L \frac{[k\gamma]}{[\tau + k\gamma]} \\ & \times \frac{[\sum_{l=1}^L (x_l - \mu_l) + (L+1)\gamma]}{[\sum_{l=1}^L (x_l - \mu_l) + \tau + (L+2)\gamma]} \det(\Omega \omega^{-1}), \end{aligned}$$

where $\omega := \Omega|_{\tau=-\gamma}$.

In order to avoid an overcrowded section we present the proof of Theorem 5.2 in Appendix A. As a matter of fact we can extract more results from the method used for proving Theorem 5.2. Once we determine the function $f(x_0, x_{\bar{0}}, x_1, \dots, x_L)$ appearing in (5.16), we automatically find another three families of determinantal representations. In order to precise the latter statement we introduce matrices

$$(5.19) \quad \Omega_i := \begin{pmatrix} \mathcal{F}_i & \mathcal{I} & \mathcal{G} \\ \bar{\mathcal{I}}_i & \mathcal{K} & \mathcal{J} \\ \bar{\mathcal{F}}_i & \bar{\mathcal{J}} & \bar{\mathcal{G}} \end{pmatrix}, \quad \bar{\Omega}_i := \begin{pmatrix} \mathcal{F} & \mathcal{I} & \mathcal{G}_i \\ \bar{\mathcal{I}} & \mathcal{K} & \mathcal{J}_i \\ \bar{\mathcal{F}} & \bar{\mathcal{J}} & \bar{\mathcal{G}}_i \end{pmatrix} \quad \text{and} \quad \tilde{\Omega}_{ij} := \begin{pmatrix} \mathcal{F} & \mathcal{I}_{ij} & \mathcal{G} \\ \bar{\mathcal{I}} & \mathcal{K}_{ij} & \mathcal{J} \\ \bar{\mathcal{F}} & \bar{\mathcal{J}}_{ij} & \bar{\mathcal{G}} \end{pmatrix}.$$

The entries of Ω_i , $\bar{\Omega}_i$ and $\tilde{\Omega}_{ij}$ have been defined in (5.11)-(5.15) and their structure are not too different from the ones constituting Ω . In this way, taking into account (5.19), we can state the following theorem.

Theorem 5.3. *Write $\omega_i := \Omega_i|_{\tau=-\gamma}$ and $\bar{\omega}_i := \bar{\Omega}_i|_{\tau=-\gamma}$ for $1 \leq i \leq L$. Also, let us define $\tilde{\omega}_{ij} := \tilde{\Omega}_{ij}|_{\tau=-\gamma}$ on the interval $1 \leq i < j \leq L$. Then there exist families of continuous determinantal representations for Z_τ given by*

$$(5.20) \quad \begin{aligned} Z_\tau(X_i^0) = & (-1)^L \left(\frac{[(L+1)\gamma]}{[\tau + (L+2)\gamma]} \right)^{d_L-1} \left(\frac{[\tau + (L+1)\gamma]}{[L\gamma]} \right)^{d_L} \prod_{x \in X_i^0} \prod_{j=1}^L [x - \mu_j] \prod_{k=1}^L \frac{[k\gamma]}{[\tau + k\gamma]} \\ & \times \frac{[\sum_{l=1}^L (x_l - \mu_l) + (L+1)\gamma]}{[\sum_{l=1}^L (x_l - \mu_l) + \tau + (L+2)\gamma]} \det(\Omega_i \omega_i^{-1}), \end{aligned}$$

$$\begin{aligned}
Z_\tau(X_i^{\bar{0}}) &= (-1)^L \left(\frac{[(L+1)\gamma]}{[\tau + (L+2)\gamma]} \right)^{d_{L-1}} \left(\frac{[\tau + (L+1)\gamma]}{[L\gamma]} \right)^{d_L} \prod_{x \in X_i^{\bar{0}}} \prod_{j=1}^L [x - \mu_j] \prod_{k=1}^L \frac{[k\gamma]}{[\tau + k\gamma]} \\
(5.21) \quad &\times \frac{[\sum_{l=1}^L (x_l - \mu_l) + (L+1)\gamma]}{[\sum_{l=1}^L (x_l - \mu_l) + \tau + (L+2)\gamma]} \det(\bar{\Omega}_i \bar{\omega}_i^{-1}) ,
\end{aligned}$$

$$\begin{aligned}
Z_\tau(X_{i,j}^{0,\bar{0}}) &= (-1)^L \left(\frac{[(L+1)\gamma]}{[\tau + (L+2)\gamma]} \right)^{d_{L-1}} \left(\frac{[\tau + (L+1)\gamma]}{[L\gamma]} \right)^{d_L} \prod_{x \in X_{i,j}^{0,\bar{0}}} \prod_{k=1}^L [x - \mu_k] \prod_{l=1}^L \frac{[l\gamma]}{[\tau + l\gamma]} \\
(5.22) \quad &\times \frac{[\sum_{m=1}^L (x_m - \mu_m) + (L+1)\gamma]}{[\sum_{m=1}^L (x_m - \mu_m) + \tau + (L+2)\gamma]} \det(\tilde{\Omega}_{i,j} \tilde{\omega}_{i,j}^{-1}) .
\end{aligned}$$

Remark 5.4. Using $\Pi_{0,i} Z_\tau(X_i^0) = Z_\tau(X)$ one can compare formulae (5.18) and (5.20). By doing so we find (5.18) and (5.20) indeed consist of different representations. Similarly, one reaches the same conclusion for representations (5.21) and (5.22).

Remark 5.5. Formulae (5.20) and (5.21) encloses L representations each since $1 \leq i \leq L$. On the other hand, (5.22) yields $L(L-1)/2$ representations due to the condition $1 \leq i, j \leq L$. All together they add up to $L(L+3)/2$ families of representations.

The proof of Theorem 5.3 can also be found in Appendix A and now let us focus again on the representation (5.18). The non-trivial part is contained in the matrix Ω (5.17) whose entries are defined in (5.7)-(5.10). In order to make our results more explicit we have also collected (5.7)-(5.15) directly in terms of theta-functions in Appendix C. From (5.7)-(5.10) one can see the entries of the matrices Ω and ω depends on the variables x_0 and $x_{\bar{0}}$, in addition to X . However, the combination $\det(\Omega \omega^{-1})$ is independent of x_0 and $x_{\bar{0}}$, and it essentially gives the partition function $Z_\tau(X)$. In this way, those two exceeding variables can be chosen at convenience without affecting the partition function. The same analysis holds for (5.20)-(5.22) where each representation also possess two extra local variables having no global influence. Therefore, both Theorems 5.2 and 5.3 give us continuous families of single determinantal representations and one can regard those extra variables as their parameterization.

Remark 5.6. The representation recently presented in [Gal16a] consists of (5.18) under specialization $x_0 = \mu_1 - 2\gamma$ and $x_{\bar{0}} = \mu_1 - \gamma$.

6. THE SIX-VERTEX MODEL LIMIT

In this section we investigate a particular limit of the partition function (2.15) where it reduces to that of the six-vertex model with domain-wall boundaries introduced in [Kor82]. This is achieved in the limit $p \rightarrow 0$ followed by the limit $\tau \rightarrow \infty$. As far as the allowed lattice configurations are concerned, it was firstly pointed out by Lenard the existence of an equivalence between SOS and vertex models configurations. The specialization of the statistical weights requires a more careful analysis and for that we employ the identity

$\lim_{p \rightarrow 0} -ip^{-\frac{1}{4}}[x] = \sinh(x)$. Moreover, in this section we also consider the limit $\tau \rightarrow \infty$ in such a way that (2.8) reduces to the standard six-vertex model weights

$$(6.1) \quad \begin{aligned} a(x) &:= \sinh(x + \gamma) \\ b(x) &:= \sinh(x) \\ c(x) &:= \sinh(\gamma) . \end{aligned}$$

In writing (6.1) we have ignored overall normalization factors. Also, as we are taking the limit $\tau \rightarrow \infty$, we simply use $Z_\infty(X) = Z(X)$ to denote the model's partition function.

The structure of functional relations type A and D, given respectively by (3.6) and (3.10), exhibit important simplifications in the six-vertex model limit. This limit eliminates the *dynamical* aspect of both equations, making the use of Eq. type AD (4.1) unnecessary. Here we shall refer to (3.6) in the six-vertex model limit as *reduced type A* equation and, analogously, (3.10) will be *reduced type D*. Moreover, both reduced type A and reduced type D equations are independently able to fix the partition function Z up to an overall multiplicative factor. We shall tackle the resolution of these equations in what follows.

6.1. Reduced type A. The derivation of Eq. (3.6) heavily relies on the dynamical Yang-Baxter algebra $\mathcal{A}(\mathcal{R})$. The latter reduces to the standard Yang-Baxter algebra associated to the six-vertex model in the limit previously discussed. Therefore, we do not lose any relevant information by taking the six-vertex model limit directly on (3.6). In fact, the functional relation (3.6) unfolds into the system (4.3) by means of permutations; and the system of equations recasted as (4.3) survives entirely in the six-vertex model limit. The latter feature can be recasted as the following corollary.

Corollary 6.1. *The partition function of the six-vertex model with domain-wall boundaries satisfies the system of equations*

$$(6.2) \quad \sum_{i=0}^L \sigma_i^{(l)} Z(X_i^0) = 0 \quad 0 \leq l \leq L ,$$

with coefficients explicitly defined as

$$\sigma_i^{(l)} := \begin{cases} \frac{c(x_0 - x_l)}{b(x_0 - x_l)} \prod_{k=1}^L a(x_0 - \mu_k) \prod_{\substack{k=1 \\ k \neq l}}^L \frac{a(x_k - x_0)}{b(x_k - x_0)} & i = 0, l \neq 0 \\ \prod_{k=1}^L b(x_l - \mu_k) - \prod_{k=1}^L a(x_l - \mu_k) \prod_{\substack{k=0 \\ k \neq l}}^L \frac{a(x_k - x_l)}{b(x_k - x_l)} & i = l \\ \frac{c(x_i - x_l)}{b(x_i - x_l)} \prod_{k=1}^L a(x_i - \mu_k) \prod_{\substack{k=0 \\ k \neq i, l}}^L \frac{a(x_k - x_i)}{b(x_k - x_i)} & \text{otherwise} \end{cases} .$$

Proof. Straightforward evaluation of the limits $p \rightarrow 0$ and $\tau \rightarrow \infty$ in (4.3). \square

Remark 6.2. The system (6.2) encloses $L + 1$ equations in consonance with the $L + 1$ terms $Z(X_i^0)$ present in each equation. Moreover, one can also verify that $\det(\sigma_i^{(l)})_{0 \leq i, l \leq L} = 0$ ensuring the system has a non-trivial solution.

The resolution of (6.2) can be performed along the lines described in Section 5. For that we single out a subset of (6.2) containing L equations, namely the ones on $1 \leq l \leq L$. This subset allows one to write each function $Z(X_i^0)$ for $1 \leq i \leq L$ in terms of $Z(X)$. By doing so we find

$$(6.3) \quad Z(X_i^0) = \frac{\det(V_i)}{\det(V)} Z(X)$$

where V and V_i are $L \times L$ matrices with entries defined as follows,

$$\begin{aligned} V_{\alpha, \beta} &:= \sigma_{\beta}^{(\alpha)} & 1 \leq \alpha, \beta \leq L \\ (V_i)_{\alpha, \beta} &:= \begin{cases} -\sigma_0^{(\alpha)} & \beta = i \\ \sigma_{\beta}^{(\alpha)} & \text{otherwise} \end{cases} \end{aligned}$$

As discussed in Section 5.1, one can regard Eq. (6.3) as an one-variable functional equation and employ separation of variables. In this way we can conclude that $\det(V) = Z(X)f(x_0, x_1, \dots, x_L)$ and $\det(V_i) = Z(X_i^0)f(x_0, x_1, \dots, x_L)$ for a given function f . Hence the full characterization of the partition function Z only requires the determination of the function f .

Theorem 6.3. *The partition function Z can be written as*

$$(6.4) \quad Z(X) = \det(V) \prod_{k=1}^L \frac{b(x_k - x_0)}{a(x_0 - \mu_k)} \quad \text{and} \quad Z(X_i^0) = \det(V_i) \prod_{k=1}^L \frac{b(x_k - x_0)}{a(x_0 - \mu_k)}$$

where

$$\begin{aligned} V_{\alpha, \beta} &= \begin{cases} \prod_{k=1}^L b(x_{\alpha} - \mu_k) - \prod_{k=1}^L a(x_{\alpha} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha}}^L \frac{a(x_k - x_{\alpha})}{b(x_k - x_{\alpha})} & \beta = \alpha \\ \frac{c(x_{\beta} - x_{\alpha})}{b(x_{\beta} - x_{\alpha})} \prod_{k=1}^L a(x_{\beta} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha, \beta}}^L \frac{a(x_k - x_{\beta})}{b(x_k - x_{\beta})} & \text{otherwise} \end{cases}, \\ (V_i)_{\alpha, \beta} &= \begin{cases} \frac{c(x_{\alpha} - x_0)}{b(x_{\alpha} - x_0)} \prod_{k=1}^L a(x_0 - \mu_k) \prod_{\substack{k=1 \\ k \neq \alpha}}^L \frac{a(x_k - x_0)}{b(x_k - x_0)} & \beta = i, 1 \leq \alpha \leq L \\ \prod_{k=1}^L b(x_{\alpha} - \mu_k) - \prod_{k=1}^L a(x_{\alpha} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha}}^L \frac{a(x_k - x_{\alpha})}{b(x_k - x_{\alpha})} & \beta = \alpha \neq i \\ \frac{c(x_{\beta} - x_{\alpha})}{b(x_{\beta} - x_{\alpha})} \prod_{k=1}^L a(x_{\beta} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha, \beta}}^L \frac{a(x_k - x_{\beta})}{b(x_k - x_{\beta})} & \text{otherwise} \end{cases}. \end{aligned}$$

The proof of Theorem 6.3 can be found in Appendix B and some comments concerning formulae (6.4) are important at this stage. For instance, the second expression of (6.4) can be matched with the first one upon the identification $\Pi_{0,i}Z(X_i^0) = Z(X)$. However, one can notice $\Pi_{0,i}V_i$ does not seem to be related to V by simple transformations. This indicates they indeed constitute independent representations. In addition to that, one of the most important aspects of both formulae in (6.4) is that they consist of continuous families of representations. For example, let us take a closer look at the first expression of (6.4). The LHS depends on variables X while the matrix entries in the RHS depends on $X \cup \{x_0\}$. Thus the dependence of the RHS with x_0 is local but not global; and x_0 can be regarded as a variable parameterizing this continuous family of representations. This feature of representations (6.4) might have important consequences as far as applications are concerned. For instance, depending on the application we have in mind, the variable x_0 can be suitably tuned in order to simplify calculations. This same argument holds for the second formula of (6.4) but now with variable x_i parameterizing the family of representations.

6.2. Reduced type D. We continue our analysis of the six-vertex model limit along the same lines described in Section 6.1. For that we firstly extend (3.10) to a system of functional relations through permutations $\Pi_{0,m}$. The six-vertex model limit is then formalized as the following corollary.

Corollary 6.4. *The system of equations*

$$(6.5) \quad \sum_{i=0}^L \rho_i^{(m)} Z(X_i^0) = 0 \quad 0 \leq m \leq L ,$$

with coefficients

$$\rho_i^{(m)} := \begin{cases} \frac{c(x_m - x_0)}{b(x_m - x_0)} \prod_{k=1}^L b(x_0 - \mu_k) \prod_{\substack{k=1 \\ k \neq m}}^L \frac{a(x_0 - x_k)}{b(x_0 - x_k)} & i = 0, m \neq 0 \\ \prod_{k=1}^L a(x_m - \mu_k) - \prod_{k=1}^L b(x_m - \mu_k) \prod_{\substack{k=0 \\ k \neq m}}^L \frac{a(x_m - x_k)}{b(x_m - x_k)} & i = m \\ \frac{c(x_m - x_i)}{b(x_m - x_i)} \prod_{k=1}^L b(x_i - \mu_k) \prod_{\substack{k=0 \\ k \neq i, m}}^L \frac{a(x_i - x_k)}{b(x_i - x_k)} & \text{otherwise} \end{cases} .$$

is satisfied by the partition function of the six-vertex model with domain-wall boundaries.

Proof. We firstly apply $\Pi_{0,m}$ onto (3.10) and then take the limits $p \rightarrow 0$ and $\tau \rightarrow \infty$. \square

Remark 6.5. The label (m) in (6.5) takes values on the interval $0 \leq m \leq L$ and thus (6.5) consists of a system of $L + 1$ equations. Also, (6.5) relates $L + 1$ unknowns of the form $Z(X_i^0)$, and the existence of non-trivial solutions requires $\det(\rho_i^{(m)})_{0 \leq i, m \leq L} = 0$. The latter condition can be readily verified.

Remark 6.6. In contrast to (3.10), we have used x_0 instead of $x_{\bar{0}}$ since this distinction is not required for the analysis of (6.5).

Next we would like to solve the system of equations (6.5) and this can be accomplished using the methodology described in Section 5 and Section 6.1. For that we focus on the subset of (6.5) formed by $1 \leq m \leq L$ which allows one to express each function $Z(X_i^0)$ in terms of $Z(X)$. This possibility is a direct consequence of the linearity of our equations. Using Cramer's rule we then find relations of the form

$$(6.6) \quad Z(X_i^0) = \frac{\det(W_i)}{\det(W)} Z(X) ,$$

The matrices W and W_i in (6.6) have dimension $L \times L$ and their entries are given in terms of the coefficients of (6.5). More precisely, they are defined as

$$\begin{aligned} W_{\alpha,\beta} &:= \rho_{\beta}^{(\alpha)} & 1 \leq \alpha, \beta \leq L \\ (W_i)_{\alpha,\beta} &:= \begin{cases} -\rho_0^{(\alpha)} & \beta = i \\ \rho_{\beta}^{(\alpha)} & \text{otherwise} \end{cases} . \end{aligned}$$

As previously discussed, one can regard (6.6) as a system of one-variable functional equations which can be solved using separation of variables. The resolution of such equations produces the following theorem.

Theorem 6.7. *The partition function Z admits the following families of continuous representations*

$$(6.7) \quad Z(X) = \det(W) \prod_{k=1}^L \frac{b(x_0 - x_k)}{b(x_0 - \mu_k)} \quad \text{and} \quad Z(X_i^0) = \det(W_i) \prod_{k=1}^L \frac{b(x_0 - x_k)}{b(x_0 - \mu_k)}$$

where

$$\begin{aligned} W_{\alpha,\beta} &= \begin{cases} \prod_{k=1}^L a(x_{\alpha} - \mu_k) - \prod_{k=1}^L b(x_{\alpha} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha}}^L \frac{a(x_{\alpha} - x_k)}{b(x_{\alpha} - x_k)} & \beta = \alpha \\ \frac{c(x_{\alpha} - x_{\beta})}{b(x_{\alpha} - x_{\beta})} \prod_{k=1}^L b(x_{\beta} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha, \beta}}^L \frac{a(x_{\beta} - x_k)}{b(x_{\beta} - x_k)} & \text{otherwise} \end{cases} , \\ (W_i)_{\alpha,\beta} &= \begin{cases} \frac{c(x_0 - x_{\alpha})}{b(x_0 - x_{\alpha})} \prod_{k=1}^L b(x_0 - \mu_k) \prod_{\substack{k=1 \\ k \neq \alpha}}^L \frac{a(x_0 - x_k)}{b(x_0 - x_k)} & \beta = i, 1 \leq \alpha \leq L \\ \prod_{k=1}^L a(x_{\alpha} - \mu_k) - \prod_{k=1}^L b(x_{\alpha} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha}}^L \frac{a(x_{\alpha} - x_k)}{b(x_{\alpha} - x_k)} & \beta = \alpha \neq i \\ \frac{c(x_{\alpha} - x_{\beta})}{b(x_{\alpha} - x_{\beta})} \prod_{k=1}^L b(x_{\beta} - \mu_k) \prod_{\substack{k=0 \\ k \neq \alpha, \beta}}^L \frac{a(x_{\beta} - x_k)}{b(x_{\beta} - x_k)} & \text{otherwise} \end{cases} . \end{aligned}$$

The proof of Theorem 6.7 is also given in Appendix B. It is important to remark here that formulae (6.7) consist of $L+1$ independent families of continuous representations with properties similar to the ones pointed out for (6.4) in Section 6.1. Although we shall not discuss those properties again in this section, we remark that altogether formulae (6.4) and (6.7) totals $2(L+1)$ families of continuous representations for the partition function Z .

APPENDIX A. PROOFS OF THEOREMS 5.2 AND 5.3

The proof of Theorems 5.2 and 5.3 follows from the determination of the function f appearing in (5.16). For that we firstly notice that relations (5.16) imply the following system of partial differential equations,

$$(A.1) \quad \begin{aligned} \frac{\partial}{\partial x_0} \left(\frac{f}{\det(\Omega)} \right) &= 0 & \frac{\partial}{\partial x_{\bar{0}}} \left(\frac{f}{\det(\Omega)} \right) &= 0 \\ \frac{\partial}{\partial x_i} \left(\frac{f}{\det(\Omega_i)} \right) &= 0 & \frac{\partial}{\partial x_{\bar{0}}} \left(\frac{f}{\det(\Omega_i)} \right) &= 0 \\ \frac{\partial}{\partial x_i} \left(\frac{f}{\det(\bar{\Omega}_i)} \right) &= 0 & \frac{\partial}{\partial x_0} \left(\frac{f}{\det(\bar{\Omega}_i)} \right) &= 0 \\ \frac{\partial}{\partial x_i} \left(\frac{f}{\det(\tilde{\Omega}_{ij})} \right) &= 0 & \frac{\partial}{\partial x_j} \left(\frac{f}{\det(\tilde{\Omega}_{ij})} \right) &= 0. \end{aligned}$$

Next we consider (5.5) and (5.6) under a particular specialization of the dynamical parameter τ . More precisely, for $\tau = -\gamma$ we find

$$(A.2) \quad \begin{aligned} \left. \frac{\det(\Omega_i)}{\det(\Omega)} \right|_{\tau=-\gamma} &= \left. \frac{Z_\tau(X_i^0)}{Z_\tau(X)} \right|_{\tau=-\gamma} = \prod_{k=1}^L \frac{[x_0 - \mu_k]}{[x_i - \mu_k]} \\ \left. \frac{\det(\bar{\Omega}_i)}{\det(\Omega)} \right|_{\tau=-\gamma} &= \left. \frac{Z_\tau(X_i^{\bar{0}})}{Z_\tau(X)} \right|_{\tau=-\gamma} = \prod_{k=1}^L \frac{[x_{\bar{0}} - \mu_k]}{[x_i - \mu_k]} \\ \left. \frac{\det(\tilde{\Omega}_{ij})}{\det(\Omega)} \right|_{\tau=-\gamma} &= \left. \frac{Z_\tau(X_{i,j}^{0,\bar{0}})}{Z_\tau(X)} \right|_{\tau=-\gamma} = \prod_{k=1}^L \frac{[x_0 - \mu_k][x_{\bar{0}} - \mu_k]}{[x_i - \mu_k][x_j - \mu_k]}. \end{aligned}$$

Equations (A.2) can now be easily solved yielding

$$(A.3) \quad Z_\tau(X)|_{\tau=-\gamma} = \mathcal{C}_1 \prod_{i,j=1}^L [x_i - \mu_j]$$

with \mathcal{C}_1 being an X -independent term. Now we look at (5.16) under the same specialization taking into account solution (A.3). This gives us the following relations:

$$\begin{aligned}
 f|_{\tau=-\gamma} &= \frac{\mathcal{C}_1^{-1} \det(\Omega)|_{\tau=-\gamma}}{\prod_{x \in X} \prod_{k=1}^L [x - \mu_k]} = \frac{\mathcal{C}_1^{-1} \det(\Omega_i)|_{\tau=-\gamma}}{\prod_{x \in X_i^0} \prod_{k=1}^L [x - \mu_k]} \\
 &= \frac{\mathcal{C}_1^{-1} \det(\bar{\Omega}_i)|_{\tau=-\gamma}}{\prod_{x \in X_i^{\bar{0}}} \prod_{k=1}^L [x - \mu_k]} = \frac{\mathcal{C}_1^{-1} \det(\tilde{\Omega}_{ij})|_{\tau=-\gamma}}{\prod_{x \in X_{i,j}^{0,\bar{0}}} \prod_{k=1}^L [x - \mu_k]} .
 \end{aligned}
 \tag{A.4}$$

Motivated by (A.4) we also consider the redefinition

$$f(x_0, x_{\bar{0}}, x_1, \dots, x_L) =: f|_{\tau=-\gamma} f_r(x_0, x_{\bar{0}}, x_1, \dots, x_L) .$$

In addition to that we also introduce matrices T , T_i , \bar{T}_i and \tilde{T}_{ij} such that

$$\begin{aligned}
 \det(T) &= \prod_{x \in X} \prod_{k=1}^L [x - \mu_k] \frac{\det(\Omega)}{\det(\Omega)|_{\tau=-\gamma}} & \det(T_i) &= \prod_{x \in X_i^0} \prod_{k=1}^L [x - \mu_k] \frac{\det(\Omega_i)}{\det(\Omega_i)|_{\tau=-\gamma}} \\
 \det(\bar{T}_i) &= \prod_{x \in X_i^{\bar{0}}} \prod_{k=1}^L [x - \mu_k] \frac{\det(\bar{\Omega}_i)}{\det(\bar{\Omega}_i)|_{\tau=-\gamma}} & \det(\tilde{T}_{ij}) &= \prod_{x \in X_{i,j}^{0,\bar{0}}} \prod_{k=1}^L [x - \mu_k] \frac{\det(\tilde{\Omega}_{ij})}{\det(\tilde{\Omega}_{ij})|_{\tau=-\gamma}} .
 \end{aligned}
 \tag{A.5}$$

Hence, we can rewrite the system of equations (A.1) as

$$(A.6) \quad \frac{\partial}{\partial x_0} \left(\frac{f_r}{\det(T)} \right) = 0 \quad (A.10) \quad \frac{\partial}{\partial x_{\bar{0}}} \left(\frac{f_r}{\det(T)} \right) = 0$$

$$(A.7) \quad \frac{\partial}{\partial x_i} \left(\frac{f_r}{\det(T_i)} \right) = 0 \quad (A.11) \quad \frac{\partial}{\partial x_{\bar{0}}} \left(\frac{f_r}{\det(T_i)} \right) = 0$$

$$(A.8) \quad \frac{\partial}{\partial x_i} \left(\frac{f_r}{\det(\bar{T}_i)} \right) = 0 \quad (A.12) \quad \frac{\partial}{\partial x_0} \left(\frac{f_r}{\det(\bar{T}_i)} \right) = 0$$

$$(A.9) \quad \frac{\partial}{\partial x_i} \left(\frac{f_r}{\det(\tilde{T}_{ij})} \right) = 0 \quad (A.13) \quad \frac{\partial}{\partial x_j} \left(\frac{f_r}{\det(\tilde{T}_{ij})} \right) = 0 .$$

Equations (A.6), (A.10), (A.11) and (A.12) simplifies to

$$(A.14) \quad \frac{\partial \log(f_r)}{\partial x_0} = \frac{\partial \log(f_r)}{\partial x_{\bar{0}}} = 0 .$$

Therefore we can conclude that $f_r(x_0, x_{\bar{0}}, x_1, \dots, x_L) = f_r(x_1, \dots, x_L)$. On the other hand, Eqs. (A.7), (A.8), (A.9) and (A.13) gives

$$(A.15) \quad \begin{aligned} \frac{\partial \log(f_r)}{\partial x_i} &= \frac{\partial}{\partial x_i} \log \left(\left[\sum_{k=1}^L (x_k - \mu_k) + \tau + (L+2)\gamma \right] \right) \\ &\quad - \frac{\partial}{\partial x_i} \log \left(\left[\sum_{k=1}^L (x_k - \mu_k) + (L+1)\gamma \right] \right), \end{aligned}$$

which can be readily integrated. In this way we find

$$(A.16) \quad f_r(X) = \mathcal{C}_0 \frac{\left[\sum_{k=1}^L (x_k - \mu_k) + \tau + (L+2)\gamma \right]}{\left[\sum_{k=1}^L (x_k - \mu_k) + (L+1)\gamma \right]}.$$

The combination of (5.16), (A.4) and (A.16) leaves us with the following expressions,

$$(A.17) \quad \begin{aligned} Z_\tau(X) &= \frac{\mathcal{C}_1}{\mathcal{C}_0} \frac{\left[\sum_{k=1}^L (x_k - \mu_k) + (L+1)\gamma \right]}{\left[\sum_{k=1}^L (x_k - \mu_k) + \tau + (L+2)\gamma \right]} \frac{\det(\Omega)}{\det(\Omega)|_{\tau=-\gamma}} \prod_{x \in X} \prod_{k=1}^L [x - u_k] \\ Z_\tau(X_i^0) &= \frac{\mathcal{C}_1}{\mathcal{C}_0} \frac{\left[\sum_{k=1}^L (x_k - \mu_k) + (L+1)\gamma \right]}{\left[\sum_{k=1}^L (x_k - \mu_k) + \tau + (L+2)\gamma \right]} \frac{\det(\Omega_i)}{\det(\Omega_i)|_{\tau=-\gamma}} \prod_{x \in X_i^0} \prod_{k=1}^L [x - u_k] \\ Z_\tau(X_i^{\bar{0}}) &= \frac{\mathcal{C}_1}{\mathcal{C}_0} \frac{\left[\sum_{k=1}^L (x_k - \mu_k) + (L+1)\gamma \right]}{\left[\sum_{k=1}^L (x_k - \mu_k) + \tau + (L+2)\gamma \right]} \frac{\det(\bar{\Omega}_i)}{\det(\bar{\Omega}_i)|_{\tau=-\gamma}} \prod_{x \in X_i^{\bar{0}}} \prod_{k=1}^L [x - u_k] \\ Z_\tau(X_{i,j}^{0,\bar{0}}) &= \frac{\mathcal{C}_1}{\mathcal{C}_0} \frac{\left[\sum_{k=1}^L (x_k - \mu_k) + (L+1)\gamma \right]}{\left[\sum_{k=1}^L (x_k - \mu_k) + \tau + (L+2)\gamma \right]} \frac{\det(\tilde{\Omega}_{ij})}{\det(\tilde{\Omega}_{ij})|_{\tau=-\gamma}} \prod_{x \in X_{i,j}^{0,\bar{0}}} \prod_{k=1}^L [x - u_k], \end{aligned}$$

and to complete the proofs of Theorems 5.2 and 5.3 we only need to determine the constant factor $\mathcal{C}_1/\mathcal{C}_0$. The latter can be obtained from the asymptotic behavior derived in [Gal13b]. By doing so we find

$$(A.18) \quad \frac{\mathcal{C}_1}{\mathcal{C}_0} = (-1)^L \left(\frac{[(L+1)\gamma]}{[\tau + (L+2)\gamma]} \right)^{d_L-1} \left(\frac{[\tau + (L+1)\gamma]}{[L\gamma]} \right)^{d_L} \prod_{k=1}^L \frac{[k\gamma]}{[\tau + k\gamma]}.$$

APPENDIX B. PROOFS OF THEOREMS 6.3 AND 6.7

The proof of Theorem 6.3 can be obtained from the analysis of relation (6.3). This relation allows one to conclude that

$$(B.1) \quad \begin{aligned} \det(V_i) &= Z(X_i^0) f(x_0, x_1, \dots, x_L) \\ \det(V) &= Z(X) f(x_0, x_1, \dots, x_L), \end{aligned}$$

and formulae (6.4) follows from the determination of the function f . In order to determine such function we notice that decomposition (B.1) induces a system of partial differential equations, namely

$$(B.2) \quad \frac{\partial}{\partial x_i} \left(\frac{f}{\det(V_i)} \right) = 0$$

$$(B.3) \quad \frac{\partial}{\partial x_0} \left(\frac{f}{\det(V)} \right) = 0 ,$$

which can be solved for f . The solution of (B.2) and (B.3) can be obtained using elementary methods and we start by noticing that (B.2) simplifies to

$$(B.4) \quad \frac{\partial \log(f)}{\partial x_i} = \frac{1}{\tanh(x_0 - x_i)} .$$

The solution of (B.4) is then given by

$$(B.5) \quad \log(f) = -\log \left(\prod_{k=1}^L b(x_0 - x_k) \right) + h(x_0) ,$$

where h is an unknown function depending on the single variable x_0 . Next we substitute (B.5) in (B.3) and this procedure yields the following constraint on h ,

$$(B.6) \quad \frac{\partial h}{\partial x_0} = \sum_{k=1}^L \frac{1}{\tanh(x_0 - \mu_k + \gamma)} .$$

Eq. (B.6) can be easily integrated and we find

$$(B.7) \quad h(x_0) = \log \left(\prod_{k=1}^L a(x_0 - \mu_k) \right) + \log(\mathcal{C})$$

where \mathcal{C} is an integration constant. Gathering (B.5) and (B.7) we obtain

$$(B.8) \quad f = \mathcal{C} \prod_{k=1}^L \frac{a(x_0 - \mu_k)}{b(x_0 - x_k)} ,$$

thus reducing our problem to the determination of \mathcal{C} . Using the asymptotic behavior derived in [Gal10] we obtain $\mathcal{C} = (-1)^L$ which completes the proof of Theorem 6.3.

Next we detail the proof of Theorem 6.7. It is analogous to the one presented for Theorem 6.3 and we start our analysis from relation (6.6). The latter allows one to write

$$(B.9) \quad \begin{aligned} \det(W_i) &= Z(X_i^0) \bar{f}(x_0, x_1, \dots, x_L) \\ \det(W) &= Z(X) \bar{f}(x_0, x_1, \dots, x_L) , \end{aligned}$$

and our task becomes the determination of the function \bar{f} . For that we consider the following system of partial differential equations,

$$(B.10) \quad \frac{\partial}{\partial x_i} \left(\frac{\bar{f}}{\det(W_i)} \right) = 0$$

$$(B.11) \quad \frac{\partial}{\partial x_0} \left(\frac{\bar{f}}{\det(W)} \right) = 0 ,$$

which is a direct consequence of (B.9). The resolution of (B.10) yields

$$(B.12) \quad \log(\bar{f}) = -\log \left(\prod_{k=1}^L b(x_0 - x_k) \right) + \bar{h}(x_0) ,$$

where \bar{h} is an arbitrary function depending solely on x_0 . The substitution of (B.12) in (B.11) yields a constraint for the function \bar{h} . This constraint can be readily solved and we find

$$(B.13) \quad \bar{h}(x_0) = \log \left(\prod_{k=1}^L b(x_0 - \mu_k) \right) + \log(\bar{\mathcal{C}}) ,$$

where $\bar{\mathcal{C}}$ is an integration constant. The combination of (B.12) and (B.13) leaves us with the expression

$$(B.14) \quad \bar{f} = \bar{\mathcal{C}} \prod_{k=1}^L \frac{b(x_0 - \mu_k)}{b(x_0 - x_k)} .$$

Lastly, we need to determine the constant $\bar{\mathcal{C}}$ and from the asymptotic behavior presented in [Gal10] we find $\bar{\mathcal{C}} = 1$ which completes our proof.

APPENDIX C. EXPLICIT FORMULAE

In this appendix we collect explicit expressions for the matrices (5.7)-(5.15) in terms of the theta-function defined in (2.6). These expressions are required for constructing the determinantal representations (5.18) and (5.20)-(5.22). For convenience, we also introduce the generalized notation

$$(C.1) \quad X_{a_1, a_2, \dots, a_n}^{b_1, b_2, \dots, b_m} := X \cup \{x_{b_1}, x_{b_2}, \dots, x_{b_m}\} \setminus \{x_{a_1}, x_{a_2}, \dots, x_{a_n}\} .$$

As far as representation (5.18) is concerned, we need to build the matrix Ω defined in (5.17). Its entries explicitly read

$$\mathcal{F}_{a,b} = \begin{cases} \frac{[x_a - x_0 + \gamma]}{[x_a - x_0]} \prod_{x \in X_a} \frac{[x_a - x + \gamma]}{[x_a - x]} \prod_{j=1}^L \frac{[x_a - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_a - \mu_j + \gamma]} \\ \quad - \frac{[x_{\bar{0}} - x_0 + \gamma]}{[x_{\bar{0}} - x_0]} \prod_{x \in X_a} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} \prod_{j=1}^L [x_{\bar{0}} - \mu_j] & a = b \\ - \frac{[\gamma][x_a - x_b + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_b]} \prod_{j=1}^L \frac{[x_b - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_a - \mu_j + \gamma]} \prod_{x \in X_{a,b}^0} \frac{[x_b - x + \gamma]}{[x_b - x]} & \text{otherwise} \end{cases} \quad (\text{C.2})$$

$$\bar{\mathcal{F}}_{a,b} = \begin{cases} - \frac{[\gamma][x_0 - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_0 - x_{\bar{0}}]} \prod_{j=1}^L \frac{[x_{\bar{0}} - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \prod_{x \in X_a} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & a = b \\ 0 & \text{otherwise} \end{cases} \quad (\text{C.3})$$

$$\mathcal{G}_{a,b} = \begin{cases} \frac{[\gamma][x_{\bar{0}} - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_{\bar{0}} - x_0]} \prod_{j=1}^L [x_0 - \mu_j] \prod_{x \in X_a} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & a = b \\ 0 & \text{otherwise} \end{cases} \quad (\text{C.4})$$

$$\bar{\mathcal{G}}_{a,b} = \begin{cases} \frac{[x_0 - x_{\bar{0}} + \gamma]}{[x_0 - x_{\bar{0}}]} \prod_{x \in X_a} \frac{[x_0 - x + \gamma]}{[x_0 - x]} \prod_{j=1}^L \frac{[x_0 - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \\ \quad - \frac{[x_a - x_{\bar{0}} + \gamma]}{[x_a - x_{\bar{0}}]} \prod_{x \in X_a} \frac{[x_a - x + \gamma]}{[x_a - x]} \prod_{j=1}^L [x_a - \mu_j] & a = b \\ \frac{[\gamma][x_a - x_b + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_b]} \prod_{j=1}^L [x_b - \mu_j] \prod_{x \in X_{a,b}^0} \frac{[x_b - x + \gamma]}{[x_b - x]} & \text{otherwise} \end{cases} \quad (\text{C.5})$$

$$\begin{aligned}
\mathcal{I}_{a,n_{r,s}} = & \begin{cases} \frac{[\gamma][x_{\bar{0}} - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_{\bar{0}} - x_r]} \prod_{j=1}^L [x_r - \mu_j] \prod_{x \in X_{r,s}^0} \frac{[x_r - x + \gamma]}{[x_r - x]} & a = s \\ \frac{[\gamma][x_{\bar{0}} - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_{\bar{0}} - x_s]} \prod_{j=1}^L [x_s - \mu_j] \prod_{x \in X_{r,s}^0} \frac{[x_s - x + \gamma]}{[x_s - x]} & a = r \\ 0 & \text{otherwise} \end{cases} \\
\text{(C.6)} &
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{J}}_{a,n_{r,s}} = & \begin{cases} -\frac{[\gamma][x_0 - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_0 - x_r]} \prod_{j=1}^L \frac{[x_r - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \times \prod_{x \in X_{r,s}^{\bar{0}}} \frac{[x_r - x + \gamma]}{[x_r - x]} & a = s \\ -\frac{[\gamma][x_0 - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_0 - x_s]} \prod_{j=1}^L \frac{[x_s - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \times \prod_{x \in X_{r,s}^{\bar{0}}} \frac{[x_s - x + \gamma]}{[x_s - x]} & a = r \\ 0 & \text{otherwise} \end{cases} \\
\text{(C.7)} &
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{I}}_{n_{l,m},b} = & \begin{cases} \frac{[\gamma][x_m - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_{\bar{0}}]} \prod_{j=1}^L [x_{\bar{0}} - \mu_j] \prod_{x \in X_{l,m}^0} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & b = l \\ -\frac{[\gamma][x_l - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_{\bar{0}}]} \prod_{j=1}^L \frac{[x_{\bar{0}} - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \times \prod_{x \in X_{l,m}^0} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & b = m \\ 0 & \text{otherwise} \end{cases} \\
\text{(C.8)} &
\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{n_{l,m},b} = & \begin{cases} \frac{[\gamma][x_m - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_0]} \prod_{j=1}^L [x_0 - \mu_j] \prod_{x \in X_{l,m}^{\bar{0}}} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & b = l \\ -\frac{[\gamma][x_l - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_0]} \prod_{j=1}^L \frac{[x_0 - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\ \quad \times \prod_{x \in X_{l,m}^{\bar{0}}} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & b = m \\ 0 & \text{otherwise} \end{cases} \\
& \text{(C.9)}
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}_{n_{l,m},n_{r,s}} = & \begin{cases} \frac{[x_l - x_0 + \gamma][x_l - x_{\bar{0}} + \gamma]}{[x_l - x_0][x_l - x_{\bar{0}}]} \prod_{x \in X_{l,m}} \frac{[x_l - x + \gamma]}{[x_l - x]} \prod_{j=1}^L \frac{[x_l - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\ - \frac{[x_m - x_0 + \gamma][x_m - x_{\bar{0}} + \gamma]}{[x_m - x_0][x_m - x_{\bar{0}}]} \prod_{x \in X_{l,m}} \frac{[x_m - x + \gamma]}{[x_m - x]} \prod_{j=1}^L [x_m - \mu_j] & l = r, m = s \\ \frac{[\gamma][x_m - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_s]} \prod_{j=1}^L [x_s - \mu_j] \prod_{x \in X_{l,m,s}^{0,\bar{0}}} \frac{[x_s - x + \gamma]}{[x_s - x]} & l = r, m \neq s \\ -\frac{[\gamma][x_l - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_s]} \prod_{j=1}^L \frac{[x_s - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\ \quad \times \prod_{x \in X_{l,m,s}^{0,\bar{0}}} \frac{[x_s - x + \gamma]}{[x_s - x]} & l \neq s, m = r \\ \frac{[\gamma][x_m - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_r]} \prod_{j=1}^L [x_r - \mu_j] \prod_{x \in X_{l,m,r}^{0,\bar{0}}} \frac{[x_r - x + \gamma]}{[x_r - x]} & l = s, m \neq r \\ -\frac{[\gamma][x_l - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_r]} \prod_{j=1}^L \frac{[x_r - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\ \quad \times \prod_{x \in X_{l,m,r}^{0,\bar{0}}} \frac{[x_r - x + \gamma]}{[x_r - x]} & l \neq r, m = s, \\ 0 & \text{otherwise} \end{cases} \\
& \text{(C.10)}
\end{aligned}$$

On the other hand, the construction of Ω_i appearing in representation (5.20) also requires explicit expressions for three additional matrices, namely \mathcal{F}_i , $\bar{\mathcal{L}}_i$ and $\bar{\mathcal{F}}_i$. They read as

follows:

$$\begin{aligned}
 (\mathcal{F}_i)_{a,b} = & \left\{ \begin{aligned} & \frac{[\gamma][x_a - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_0]} \prod_{j=1}^L \frac{[x_0 - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_a - \mu_j + \gamma]} \\ & \quad \times \prod_{x \in X_a} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & i = b \\ & \frac{[x_a - x_0 + \gamma]}{[x_a - x_0]} \prod_{x \in X_a} \frac{[x_a - x + \gamma]}{[x_a - x]} \prod_{j=1}^L \frac{[x_a - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_a - \mu_j + \gamma]} \\ & \quad - \frac{[x_{\bar{0}} - x_0 + \gamma]}{[x_{\bar{0}} - x_0]} \prod_{x \in X_a} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} \prod_{j=1}^L [x_{\bar{0}} - \mu_j] & i \neq b, a = b \\ & - \frac{[\gamma][x_a - x_b + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_b]} \prod_{j=1}^L \frac{[x_b - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_a - \mu_j + \gamma]} \\ & \quad \times \prod_{x \in X_{a,b}^0} \frac{[x_b - x + \gamma]}{[x_b - x]} & i \neq b, a \neq b \end{aligned} \right. \\
 \text{(C.11)}
 \end{aligned}$$

$$\begin{aligned}
 (\bar{\mathcal{I}}_i)_{n_{l,m},b} = & \left\{ \begin{aligned} & 0 & i = b \\ & \frac{[\gamma][x_m - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_{\bar{0}}]} \prod_{j=1}^L [x_{\bar{0}} - \mu_j] \prod_{x \in X_{l,m}^0} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & i \neq b, b = l \\ & - \frac{[\gamma][x_l - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_{\bar{0}}]} \prod_{j=1}^L \frac{[x_{\bar{0}} - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\ & \quad \times \prod_{x \in X_{l,m}^0} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & i \neq b, b = m \\ & 0 & i \neq b, b \neq l, m \end{aligned} \right. \\
 \text{(C.12)}
 \end{aligned}$$

$$\begin{aligned}
(\bar{\mathcal{F}}_i)_{a,b} = & \begin{cases} -\frac{[\gamma][x_a - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_{\bar{0}}]} \prod_{j=1}^L [x_{\bar{0}} - \mu_j] \prod_{x \in X_a} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & i = b \\ -\frac{[\gamma][x_0 - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_0 - x_{\bar{0}}]} \prod_{j=1}^L \frac{[x_{\bar{0}} - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \\ \quad \times \prod_{x \in X_a} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & i \neq b, a = b \\ 0 & i \neq b, a \neq b \end{cases} \\
\text{(C.13)} &
\end{aligned}$$

Next we consider the additional matrices \mathcal{G}_i , \mathcal{J}_i and $\bar{\mathcal{G}}_i$ required for building representation (5.21) through the matrix $\bar{\Omega}_i$ defined in (5.19). These elements are given by:

$$\begin{aligned}
(\mathcal{G}_i)_{a,b} = & \begin{cases} \frac{[\gamma][x_a - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_0]} \prod_{j=1}^L \frac{[x_0 - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_a - \mu_j + \gamma]} \\ \quad \times \prod_{x \in X_a} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & i = b \\ \frac{[\gamma][x_{\bar{0}} - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_{\bar{0}} - x_0]} \prod_{j=1}^L [x_0 - \mu_j] \prod_{x \in X_a} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & i \neq b, a = b \\ 0 & i \neq b, a \neq b \end{cases} \\
\text{(C.14)} &
\end{aligned}$$

$$\begin{aligned}
(\mathcal{J}_i)_{n_{l,m},b} = & \begin{cases} 0 & i = b \\ \frac{[\gamma][x_m - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_0]} \prod_{j=1}^L [x_0 - \mu_j] \prod_{x \in X_{l,m}^{\bar{0}}} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & i \neq b, b = l \\ -\frac{[\gamma][x_l - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_0]} \prod_{j=1}^L \frac{[x_0 - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\ \quad \times \prod_{x \in X_{l,m}^{\bar{0}}} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & i \neq b, b = m \\ 0 & i \neq b, b \neq l, m \end{cases} \\
\text{(C.15)} &
\end{aligned}$$

$$\begin{aligned}
(\bar{\mathcal{G}}_i)_{a,b} = & \left\{ \begin{aligned} & -\frac{[\gamma][x_a - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_{\bar{0}}]} \prod_{j=1}^L [x_{\bar{0}} - \mu_j] \prod_{x \in X_a} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & i = b \\ & \frac{[x_0 - x_{\bar{0}} + \gamma]}{[x_0 - x_{\bar{0}}]} \prod_{x \in X_a} \frac{[x_0 - x + \gamma]}{[x_0 - x]} \prod_{j=1}^L \frac{[x_0 - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \\ & - \frac{[x_a - x_{\bar{0}} + \gamma]}{[x_a - x_{\bar{0}}]} \prod_{x \in X_a} \frac{[x_a - x + \gamma]}{[x_a - x]} \prod_{j=1}^L [x_a - \mu_j] & i \neq b, a = b \\ & \frac{[\gamma][x_a - x_b + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_b]} \prod_{j=1}^L [x_b - \mu_j] \prod_{x \in X_{a,b}^{\bar{0}}} \frac{[x_b - x + \gamma]}{[x_b - x]} & i \neq b, a \neq b \end{aligned} \right. \\
& \text{(C.16)}
\end{aligned}$$

Lastly, we also need explicit formulae for the matrices \mathcal{I}_{ij} , \mathcal{K}_{ij} and $\bar{\mathcal{J}}_{ij}$. They are required in order to build matrix $\tilde{\Omega}_{ij}$ and consequently representation (5.22). Their explicit expressions are the following:

$$\begin{aligned}
(\mathcal{I}_{ij})_{a,n_{r,s}} = & \left\{ \begin{aligned} & \frac{[\gamma][x_a - x_0 + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_0]} \prod_{j=1}^L \frac{[x_0 - \mu_j][x_{\bar{0}} - \mu_j + \gamma]}{[x_a - \mu_j + \gamma]} \\ & \quad \times \prod_{x \in X_a} \frac{[x_0 - x + \gamma]}{[x_0 - x]} & i = r, j = s \\ & \frac{[\gamma][x_{\bar{0}} - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_{\bar{0}} - x_r]} \prod_{j=1}^L [x_r - \mu_j] \prod_{x \in X_{r,s}^0} \frac{[x_r - x + \gamma]}{[x_r - x]} & i \neq r, j \neq s, a = s \\ & \frac{[\gamma][x_{\bar{0}} - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_{\bar{0}} - x_s]} \prod_{j=1}^L [x_s - \mu_j] \prod_{x \in X_{r,s}^0} \frac{[x_s - x + \gamma]}{[x_s - x]} & i \neq r, j \neq s, a = r \\ & 0 & i \neq r, j \neq s, a \neq r, s \end{aligned} \right. \\
& \text{(C.17)}
\end{aligned}$$

$$\begin{aligned}
& (\mathcal{K}_{ij})_{n_{l,m}, n_{r,s}} = \\
& \left\{ \begin{aligned}
& 0 && i = r, j = s \\
& \frac{[x_l - x_0 + \gamma][x_l - x_{\bar{0}} + \gamma]}{[x_l - x_0][x_l - x_{\bar{0}}]} \prod_{x \in X_{l,m}} \frac{[x_l - x + \gamma]}{[x_l - x]} \prod_{j=1}^L \frac{[x_l - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\
& - \frac{[x_m - x_0 + \gamma][x_m - x_{\bar{0}} + \gamma]}{[x_m - x_0][x_m - x_{\bar{0}}]} \prod_{x \in X_{l,m}} \frac{[x_m - x + \gamma]}{[x_m - x]} \prod_{j=1}^L [x_m - \mu_j] && \begin{matrix} i \neq r, j \neq s \\ l=r, m=s \end{matrix} \\
& \frac{[\gamma][x_m - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_s]} \prod_{j=1}^L [x_s - \mu_j] \prod_{x \in X_{l,m,s}^{0,\bar{0}}} \frac{[x_s - x + \gamma]}{[x_s - x]} && \begin{matrix} i \neq r, j \neq s \\ l=r, m \neq s \end{matrix} \\
& - \frac{[\gamma][x_l - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_s]} \prod_{j=1}^L \frac{[x_s - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\
& \quad \times \prod_{x \in X_{l,m,s}^{0,\bar{0}}} \frac{[x_s - x + \gamma]}{[x_s - x]} && \begin{matrix} i \neq r, j \neq s \\ l \neq s, m=r \end{matrix} \\
& \frac{[\gamma][x_m - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_m - x_r]} \prod_{j=1}^L [x_r - \mu_j] \prod_{x \in X_{l,m,r}^{0,\bar{0}}} \frac{[x_r - x + \gamma]}{[x_r - x]} && \begin{matrix} i \neq r, j \neq s \\ l=s, m \neq r \end{matrix} \\
& - \frac{[\gamma][x_l - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_l - x_r]} \prod_{j=1}^L \frac{[x_r - \mu_j][x_m - \mu_j + \gamma]}{[x_l - \mu_j + \gamma]} \\
& \quad \times \prod_{x \in X_{l,m,r}^{0,\bar{0}}} \frac{[x_r - x + \gamma]}{[x_r - x]} && \begin{matrix} i \neq r, j \neq s \\ l \neq r, m=s \end{matrix} \\
& 0 && \begin{matrix} i \neq r, j \neq s \\ l \neq r, s, m \neq r, s \end{matrix}
\end{aligned} \right.
\end{aligned}
\tag{C.18}$$

$$\begin{aligned}
& (\bar{\mathcal{J}}_{ij})_{a,n_{r,s}} = \\
& \left\{ \begin{aligned}
& -\frac{[\gamma][x_a - x_{\bar{0}} + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_a - x_{\bar{0}}]} \prod_{j=1}^L [x_{\bar{0}} - \mu_j] \prod_{x \in X_a} \frac{[x_{\bar{0}} - x + \gamma]}{[x_{\bar{0}} - x]} & i = r, j = s \\
& -\frac{[\gamma][x_0 - x_r + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_0 - x_r]} \prod_{j=1}^L \frac{[x_r - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \\
& \quad \times \prod_{x \in X_{r,s}^{\bar{0}}} \frac{[x_r - x + \gamma]}{[x_r - x]} & i \neq r, j \neq s, a = s \\
& -\frac{[\gamma][x_0 - x_s + \tau + (L+1)\gamma]}{[\tau + (L+1)\gamma][x_0 - x_s]} \prod_{j=1}^L \frac{[x_s - \mu_j][x_a - \mu_j + \gamma]}{[x_0 - \mu_j + \gamma]} \\
& \quad \times \prod_{x \in X_{r,s}^{\bar{0}}} \frac{[x_s - x + \gamma]}{[x_s - x]} & i \neq r, j \neq s, a = r \\
& 0 & i \neq r, j \neq s, a \neq r, s
\end{aligned} \right. .
\end{aligned}
\tag{C.19}$$

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